RANDOM VARIATE GENERATION
FOR EXPONENTIALLY AND POLYNOMIALLY
TILTED STABLE DISTRIBUTIONS

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ABSTRACT. We develop exact random variate generators for the polynomially and exponentially tilted unilateral stable distributions. The algorithms, which generalize Kanter's method, are uniformly fast over all choices of the tilting and stable parameters. The key to the solution is a new distribution which we call Zolotarev's distribution. We also present a novel double rejection method that is useful whenever densities have an integral representation involving an auxiliary variable.

KEYWORDS AND PHRASES. Random variate generation. Stable distribution. Tempered distributions. Rejection method. Importance sampling. Simulation. Monte Carlo method. Expected time analysis. Probability inequalities.

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Introduction

The unilateral stable random variable S_α of parameter $\alpha \in (0,1)$ has Laplace transform

$$\mathbf{E}\left\{e^{-\lambda S_\alpha}\right\} = e^{-\lambda^\alpha}, \lambda \geq 0.$$

Its properties are well-known, see, e.g., Zolotarev (1986). A simple random variate generator for S_α has been suggested by Kanter (1975), who used an integral representation of Zolotarev (1966) (see Zolotarev (1986, p. 74)), which states that the distribution function of $S_\alpha^{\alpha/(1-\alpha)}$ is given by

$$\frac{1}{\pi} \int_0^\pi e^{-\frac{A(u)}{x}} \, du, \tag{1}$$

where A is Zolotarev's function:

$$A(u) \stackrel{\text{def}}{=} \left\{ \frac{(\sin(\alpha u))^\alpha (\sin((1-\alpha)u)^{1-\alpha}}{\sin u} \right\}^{\frac{1}{1-\alpha}}.$$

By taking limits, we note that $S_1 = 1$, so that the family is properly defined for all $\alpha \in (0,1]$. Zolotarev's integral representation implies that

$$S_\alpha \stackrel{\mathcal{L}}{=} \left(\frac{A(U)}{E} \right)^{\frac{1-\alpha}{\alpha}}, \tag{2}$$

where U is uniform on $[0,\pi]$ and E is exponential with mean one. Here $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution. This is Kanter's method.

Since then, a similar generator has been proposed for all stable random variables by Chambers, Mallows and Stuck (1976), which was again based on Zolotarev's integral representation of stable distributions. A different method based on the series expansion of the stable density was developed by Devroye (1986). The purpose of this paper is to discuss random variate generation for tilted versions of S_α. We write g_α for the density of S_α, which, as we know, is concentrated on the positive real axis.

Following Perman, Pitman and Yor (1992) in connection with local times of generalizations of Bessel processes (see also James, 2006), we define the polynomially tilted random variable $T_{\alpha,\beta}$ as the random variable with density proportional to $x^{-\beta} g_\alpha(x), x > 0$. Here $\beta \geq 0$ is the tilting parameter. Due to the rapid inverse exponential decrease of g_α near the origin, $x^{-\beta} g_\alpha(x)$ is indeed integrable for all $\beta \geq 0$. Many properties of this family were derived by James (2006), who also points out its importance. For instance, if one generates a random partition of the integers according to a two parameter Poisson-Dirichlet distribution (also known as the Ewens-Pitman Sampling formula) with α positive, then the limiting distribution of the number of blocks (K_n) is almost surely $n^\alpha (T_{\alpha,\beta}^{-\alpha} + o(1))$ (Pitman, 2003, section 6.1). The tilted stable law, the stable law, the gamma distribution, the Linnik distribution, and Lamperti laws are all related by a distributional algebra, in which simple combinations of some members of these families yield other ones (see, e.g., James (2007) and some discussion in the next section). For example, $T_{1/2,\beta}$ gives us the family of inverse gamma random variables. We will represent $T_{\alpha,\beta}$ using a generalization of (2). However, that generalization involves a new random variable, which for reasons that will be become apparent further on, will be called a Zolotarev random variable. An efficient random variate generator for this new distribution then concludes that section.

Exponentially tilted, or Esscher-transformed (Sato, 1999), random variables have densities on the positive real axis that are multiplied with $ce^{-\lambda x}$ for some $\lambda > 0$ and a normalization constant c.

They play an important role in the importance sampling and in the study and simulation of rare events (see, e.g., Sadowsky and Bucklew (1989), and Bucklew (1990, 2004)). If X has density f and Y is the Esscher-transformed variable, then a generic rejection algorithm would keep on generating independent pairs (X, U) with U uniform $[0, 1]$ independent of X until for the first time $U \leq \exp(-\lambda X)$, and then return $Y \leftarrow X$. While exact, the expected number of iterations is $1/\mathrm{E}\{\exp(-\lambda X)\}$, which for any nonzero random variable X is unbounded in λ. It is of interest to develop methods that are uniformly fast in λ. This paper solves that exercise for the exponentially tilted stable law. We write $S_{\alpha,\lambda}$ for the random variable with density proportional to $e^{-\lambda x} g_\alpha(x), x > 0$. Note that exponential tilting does not make sense for stable variables that are supported on the real line. For this family, also called the tempered stable family by Barndorff-Nielsen and Shepard (2001), a path is followed based upon Zolotarev's integral representation. An efficient algorithm that uses the notion of a double rejection is given. The idea of exponentially tilting stable and other distributions was explored by Hougaard (1986). Many of their properties are well-known, see, e.g., Aalen (1992), Brix (1996), Barndorff-Nielsen and Shepard (2001) and Cerquetti (2007). Some additional interesting properties of the family can be found in James (2006) and James and Yor (2007). The exponentially tilted stable law is the natural exponential family for the positive stable distribution, and thus plays a key role in mathematical statistics, as a model for randomness used by Bayesians, and in economic models.

It should be stressed that the methods in this note never require the computation or approximation of the stable density or distribution function. The onus is plainly on Zolotarev's function. In fact, we believe that the Zolotarev distribution on which our methods are based will find many uses in the study of stable processes.

A final note about our algorithms and distributional identities: when we say "generate a random variate X", we mean that an independent experiment is carried out. When a distributional identity has a function F of several random variables, as in $F(X, Y, Z)$, then, unless it explicitly noted, all random variables involved are independent.

The polynomially tilted unilateral stable distribution

For a thorough treatment of this distribution, it helps to have a table of the moments of various random variables nearby. Distributions that satisfy Carleman's condition (see, e.g., Shohat and Tamarkin (1943) or Stoyanov (2000)) are uniquely defined by their moments. For most of the algebra or calculus of the gamma, beta and stable distributions, this is a convenient and underused method of proof. In what follows, $r \geq 0$ is fixed. We recall that

$$\mathrm{E}\{S_\alpha^{-r}\} = \frac{\Gamma\left(1 + \frac{r}{\alpha}\right)}{\Gamma(1 + r)}.$$

If G_α is a gamma random variable with parameter $\alpha > 0$, then

$$\mathrm{E}\{G_\alpha^r\} = \frac{\Gamma(r + \alpha)}{\Gamma(\alpha)}.$$

The density of $T_{\alpha,\beta}$ is then given by

$$\frac{\Gamma(1 + \beta)}{\Gamma\left(1 + \frac{\beta}{\alpha}\right)} x^{-\beta} g_\alpha(x), x > 0.$$

Indeed, the constant factor in the last expression is easily seen to be $1/\mathrm{E}\{S_\alpha^{-\beta}\}$, so that the integral is one. Thus, we have

$$\mathrm{E}\left\{T_{\alpha,\beta}^{-r}\right\} = \frac{\Gamma(1+\beta)}{\Gamma\left(1+\frac{\beta}{\alpha}\right)}\,\mathrm{E}\{S_\alpha^{-r-\beta}\} = \frac{\Gamma(1+\beta)\Gamma\left(1+\frac{r+\beta}{\alpha}\right)}{\Gamma\left(1+\frac{\beta}{\alpha}\right)\Gamma(1+r+\beta)}.$$

Simple moment comparisons give us a host of well-known results but also some properties first shown by James (2006):

$$\left(\frac{G_1}{S_\alpha}\right)^\alpha \stackrel{\mathcal{L}}{=} G_1, \quad \left(\frac{G_\beta}{T_{\alpha,\beta}}\right)^\alpha \stackrel{\mathcal{L}}{=} G_{\beta/\alpha},$$

and

$$T_{\alpha,\beta} \stackrel{\mathcal{L}}{=} T_{\gamma,\beta}T_{\alpha/\gamma,\beta/\gamma}^{1/\gamma}.$$

Choosing $\beta = 1$ above, we have

$$T_{\alpha,1} \stackrel{\mathcal{L}}{=} T_{\gamma,1}T_{\alpha/\gamma}^{1/\gamma}.$$

Choosing $\beta = 0$ ($T_{\alpha,0} = S_\alpha$), we have

$$S_\alpha \stackrel{\mathcal{L}}{=} S_\gamma S_{\alpha/\gamma}^{1/\gamma}.$$

To generalize Zolotarev's integral representation, we define the Zolotarev random variable $Z_{\alpha,b}$ ($\alpha \in (0,1), b \geq 0$) on $[0,\pi]$ with density given by

$$f(x) = \frac{\Gamma(1+b\alpha)\Gamma\left(1+b(1-\alpha)\right)}{\pi\Gamma\left(1+b\right)\,A(x)^{b(1-\alpha)}}.$$

We will show that f is bounded and non-increasing on its support.

THEOREM 1.

$$T_{\alpha,\beta} \stackrel{\mathcal{L}}{=} \left(\frac{A(Z_{\alpha,\beta/\alpha})}{G_{1+\beta\frac{1-\alpha}{\alpha}}}\right)^{\frac{1-\alpha}{\alpha}}.$$

Theorem 1 specializes to Kanter's method for $\beta = 0$, as $Z_{\alpha,0} \stackrel{\mathcal{L}}{=} U$, the uniform random variable on $[0,\pi]$, and G_1 is exponentially distributed. Random variate generation simply requires one Zolotarev random variable and one gamma random variable. For the gamma distribution, there are numerous methods that have expected time uniformly bounded in the parameters, see, e.g., Devroye (1986), Cheng (1977), Le Minh (1988), Marsaglia (1977), Ahrens and Dieter (1982), Cheng and Feast (1979, 1980), Schmeiser and Lal (1980), Ahrens, Kohrt and Dieter (1983), and Best (1983), to name just a few.

In the next section, we study Zolotarev's distribution and develop a random variate generator for it.

PROOF. Write the density of $T_{\alpha,\beta}$ as $Cx^{-\beta}g_\alpha(x)$. The density of $T_{\alpha,\beta}^{\alpha/(1-\alpha)}$ then is

$$C\frac{1-\alpha}{\alpha}x^{-\beta\frac{1-\alpha}{\alpha}+\frac{1}{\alpha}-2}g_\alpha(x^{\frac{1-\alpha}{\alpha}}).$$

Starting from Zolotarev's integral expression (1), we see that the density of $S_\alpha^{\alpha/(1-\alpha)}$ is

$$\frac{1}{\pi}\int_0^\pi \frac{A(u)}{x^2}e^{-\frac{A(u)}{x}}\,du.$$

By definition of g_α, this must be identical to

$$\frac{1-\alpha}{\alpha}x^{\frac{1}{\alpha}-2}g_\alpha(x^{\frac{1-\alpha}{\alpha}}).$$

Replace g_α in the density of $T_{\alpha,\beta}^{\alpha/(1-\alpha)}$, which then becomes

$$Cx^{-\beta\frac{1-\alpha}{\alpha}}\frac{1}{\pi}\int_0^\pi \frac{A(u)}{x^2}e^{-\frac{A(u)}{x}}\,du.$$

Finally, $T_{\alpha,\beta}^{-\alpha/(1-\alpha)}$ has the following density on $[0,\infty)$:

$$\frac{1}{\pi}\int_0^\pi Cx^{\beta\frac{1-\alpha}{\alpha}}A(u)e^{-A(u)x}\,du=\int_0^\pi\frac{C\Gamma\left(1+\beta\frac{1-\alpha}{\alpha}\right)}{\pi A(u)^{\beta\frac{1-\alpha}{\alpha}}}h(x,u)\,du$$

where $h(x,u)$ is the density of $G_{1+\beta\frac{1-\alpha}{\alpha}}/A(u)$, that is,

$$h(x,u)=\frac{A(u)\left(A(u)x\right)^{\beta\frac{1-\alpha}{\alpha}}e^{-A(u)x}}{\Gamma\left(1+\beta\frac{1-\alpha}{\alpha}\right)},x>0.$$

Note that the density of $T_{\alpha,\beta}^{-\alpha/(1-\alpha)}$ is thus

$$\int_0^\pi \ell(u)h(x,u)du$$

where ℓ is the density of $Z_{\alpha,\beta/\alpha}$. This concludes the proof. $\square$

Zolotarev's distribution

Summarizing the computations of the previous section, we write the density of $Z_{\alpha,b}$ explicitly as

$$f(x)=CA(x)^{-b(1-\alpha)}=CB(x)^b \stackrel{\text{def}}{=} C\left(\frac{\sin(x)}{(\sin(\alpha x))^\alpha(\sin((1-\alpha)x))^{1-\alpha}}\right)^b,0\leq x<\pi,$$

with

$$C=\frac{\Gamma(1+b\alpha)\Gamma\left(1+b(1-\alpha)\right)}{\pi\Gamma\left(1+b\right)}.$$

This family contains the uniform density on $[0,\pi]$ (case $b=0$), and is symmetric in α as $Z_{\alpha,b}\stackrel{\mathcal{L}}{=}Z_{1-\alpha,b}$. Furthermore, for fixed α, as $b\to\infty$,

$$\sqrt{b\alpha(1-\alpha)}Z_{\alpha,b}\stackrel{\mathcal{L}}{\to}|N|,$$

where N is standard normal. The basic form for $b = 1$ has a simple coefficient:

$$f(x) = \frac{\alpha(1-\alpha)}{\sin(\pi\alpha)} \frac{\sin(x)}{(\sin(\alpha x))^\alpha(\sin((1-\alpha)x))^{1-\alpha}}, 0 \le x < \pi.$$

Set $L(x) = \log B(x)$. Then

$$L'(x) = \cot x - \alpha^2 \cot(\alpha x) - (1-\alpha)^2 \cot((1-\alpha)x).$$

Recall the series expansion for the cotangent,

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots - \frac{(-1)^{n-1}2^{2n}B_{2n}x^{2n-1}}{(2n)!} - \cdots,$$

which is convergent for $0 < x < \pi$, and has all its coefficients negative except the first one. Here, B_{2n} is the $2n$-th Bernoulli number. Note in particular that on $(0, \pi)$, all its derivatives are negative. Since $L' \le 0$, the density is nonincreasing on $[0, \pi]$. Because $L'' \le 0$ on the given interval, B, and thus f, are log-concave on the given interval. Since $L'(0) = 0$ and $L''' \le 0$, we have the obvious inequality

$$B(x) \le B(0)e^{L''(0)x^2/2}.$$

In particular, $L''(0) = -(1/3)(1 - \alpha^3 - (1-\alpha)^3) = -\alpha(1-\alpha)$. Thus,

$$f(x) \le CB(0)^b e^{-\frac{b\alpha(1-\alpha)x^2}{2}} \stackrel{\text{def}}{=} CB(0)^b e^{-\frac{x^2}{2\sigma^2}}.$$

Here a simple limit argument yields the value $B(0) = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$.

Later on in the paper we also need a lower bound on $B(x)$ that is sharp near $x = \pi$. Using $x \ge \sin x \ge x(1 - \pi/x)$ (see, e.g., Abramowitz and Stegun (1970), p. 75), we have

$$B(x) \ge \frac{B(0)(\pi - x)}{\pi}, 0, x < \pi.$$

We do not know how to generate a random variate $Z_{\alpha,b}$ by the inversion method. However, in view of the gaussian upper bound given above, we can generate a Zolotarev random variate by rejection (see, e.g., Devroye (1986) for material and examples on this method, which is originally due to von Neumann (1951)). We suggest two cases. If $\sigma \ge \sqrt{2\pi}$, rejection from a uniform random variate is convenient. Otherwise, we use a normal dominating curve as suggested in the bound above. The details are given below.

```
set σ = 1/√(bα(1 − α)).

if σ ≥ √(2π):
    repeat:  Generate U uniformly on [0, π] and V uniformly on [0, 1].
             Set X ← U.
    until V B(0)^b ≤ B(X)^b
    return X

if σ < √(2π):
    repeat:  Generate N standard normal, and V uniformly on [0, 1].
             Set X ← σ|N|.
    until X ≤ π and V B(0)^b e^{−N²/2} ≤ B(X)^b
    return X
```

In the algorithm above, it is not recommended to use the inequalities as stated when b is large. Taking logarithms, and using the fact that $-\log V$ is distributed as an exponential E, it is better to check that

$$-E \leq b \log \left(\frac{B(X)}{B(0)} \right)$$

or

$$-E - \frac{N^2}{2} \leq b \log \left(\frac{B(X)}{B(0)} \right),$$

depending upon the case. To reduce numeric cancelations for small values of the argument, it is a good idea to use the sinc function instead of sin. Set $\mathcal{S}(x) = (\sin x)/x$, and note that

$$\frac{B(x)}{B(0)} = \frac{\mathcal{S}(x)}{(\mathcal{S}(\alpha x))^\alpha (\mathcal{S}((1 - \alpha)x))^{1-\alpha}}.$$

Each test in the algorithm now requires an exponential and a gaussian random variate, as well as three computations of the sinc function. In the next section, we establish that the algorithm above makes an expected number of iterations that is uniformly bounded over all values of α and b. Moreover, the expected number of loops tends to one as $b \to \infty$.
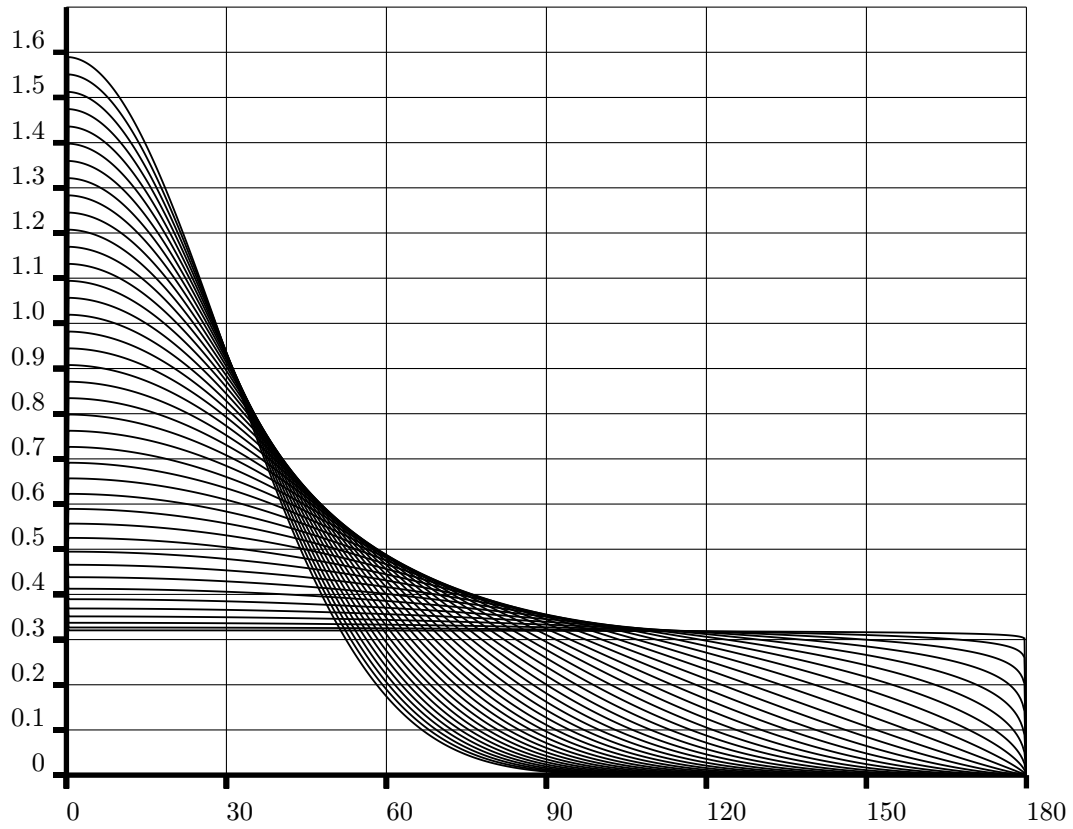
Figure 1. The Zolotarev density is plotted for $\alpha = 0.4$, with b increasing from bottom to top, taking the values $i^2/40$, $1 \le i \le 40$. The x-axis shows $[0, \pi]$ in degrees. As $b \downarrow 0$, the uniform function is reached. When $b \to \infty$, the mass concentrates at the origin, and the normal density is approached after proper rescaling.
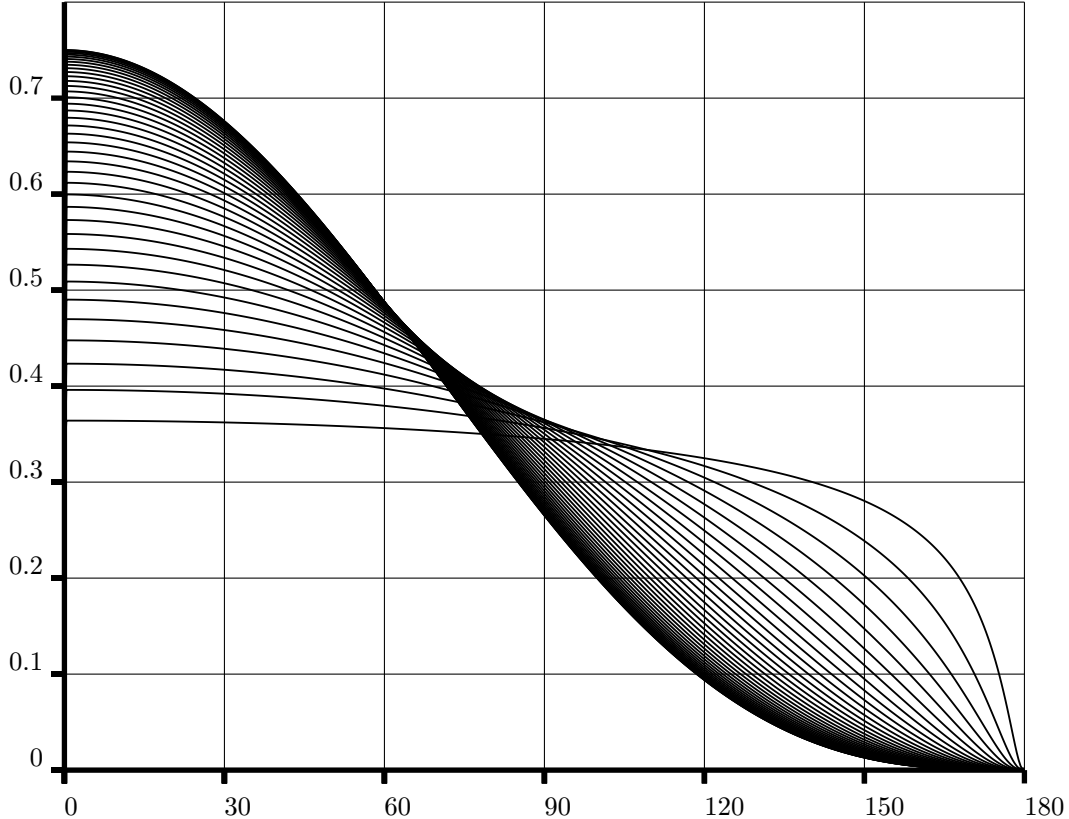
Figure 2. The Zolotarev density is plotted for $b = 3$, with α increasing from bottom to top, taking the values $i/80$, $1 \le i \le 40$. The x-axis shows $[0, \pi]$ in degrees. As $\alpha \downarrow 0$, the uniform function is reached.

REMARK 1. LOG-CONCAVITY. Devroye (1984) proposed a universal and uniformly fast rejection algorithm that works for all univariate log-concave distributions. It was subsequently improved and extended in a number of papers (Hörmann (1994), Leydold (2000, 2001)) and a book (Hörmann, Leydold and Derflinger, 2004). We could have employed that method here, but at a tremendous expense, because the method requires the explicit computation of the normalization constant, which in our case involves computing the gamma function thrice. In some cases, bounding the normalization constant can be considered, but that too makes the algorithm less elegant. The simplicity of the code, and the natural choice of a normal dominating curve have made us decide in favor of the method presented here. □

Expected time analysis

It is well-known that the expected number of iterations in the rejection method is equal to the integral under the dominating curve. In the uniform case, the integral is $CB(0)^b\pi$. In the gaussian case, it is

$$CB(0)^b \int_0^\infty e^{-\frac{x^2}{2\sigma^2}}\, dx = CB(0)^b \sigma \sqrt{\frac{\pi}{2}}.$$

Unifying both cases (and thus revealing why $\sqrt{2\pi}$ was our threshold), we see that the rejection constant is

$$R(\alpha, b) \stackrel{\text{def}}{=} CB(0)^b \pi \min\left(1, \frac{\sigma}{\sqrt{2\pi}}\right).$$

THEOREM 2. *For fixed $\alpha \in (0,1)$, $\lim_{b\to\infty} R(\alpha,b) = 1$. Furthermore,*

$$\sup_{\alpha\in(0,1),b\geq 0} R(\alpha, b) \leq \frac{e^3\sqrt{1+2\pi}}{\sqrt{4\pi}}.$$

PROOF. For the first part, we use well-known explicit bounds on Stirling's approximation, valid for all $x \geq 0$:

$$1 \leq \frac{\Gamma(1+x)}{(x/e)^x\sqrt{2\pi x}} \leq e^{\frac{1}{12x}}.$$

Then

$$C = \frac{\Gamma(1+b\alpha)\Gamma\left(1+b(1-\alpha)\right)}{\pi\Gamma\left(1+b\right)} \leq \frac{e^{\frac{1}{12b\alpha}+\frac{1}{12b(1-\alpha)}} \times \sqrt{2\pi b\alpha(1-\alpha)}}{\pi B(0)^b} = \frac{(1+o(1))\sqrt{2\pi}}{\sigma\pi B(0)^b}.$$

Thus, $R(\alpha,b) \to 1$ as $b \to \infty$.

The second part can proceed in the same manner by using

$$\sup_{x\geq 0} \frac{\Gamma(1+x)}{(x/e)^x\sqrt{2\pi(x+1)}} \leq \frac{e}{\sqrt{2\pi}}.$$

This follows quite easily from the former bound when $x \geq 1$. For $x \in [0,1]$, $\Gamma(x)$ is bounded from above by one, and $(x/e)^x\sqrt{x+1} \geq 1/e$. Also, for a lower bound, use

$$\Gamma(1+x) \geq \left(\frac{x+1}{e}\right)^{x+1}\sqrt{\frac{2\pi}{x+1}} = \frac{1}{e}\left(\frac{x+1}{e}\right)^x\sqrt{2\pi(x+1)} \geq \frac{1}{e}\left(\frac{x}{e}\right)^x\sqrt{2\pi(x+1)}.$$

Combining upper and lower bounds, we see that

$$\begin{aligned}
CB(0)^b &\leq \frac{e^3}{2\pi^2} \times \sqrt{\frac{2\pi(1+b\alpha)(1+b(1-\alpha))}{1+b}} \\
&= \frac{e^3}{\pi\sqrt{2}}\sqrt{\frac{1+b+b^2\alpha(1-\alpha))}{1+b}} \\
&\stackrel{\text{def}}{=} D\sqrt{1+\frac{b}{(1+b)\sigma^2}} \\
&\leq D\sqrt{1+\frac{1}{\sigma^2}}.
\end{aligned}$$

Thus,

$$R(\alpha, b) \leq \begin{cases} \frac{D\pi\sqrt{1+2\pi}}{\sqrt{2\pi}} & \text{when } \sigma \leq \sqrt{2\pi} \text{ ;} \\ D\pi\sqrt{1+1/(2\pi)} & \text{when } \sigma \geq \sqrt{2\pi} \text{ .} \end{cases} \quad \square$$

REMARK 2. The second part of Theorem implies that as $b \to \infty$ for α held fixed,

$$\sqrt{b\alpha(1-\alpha)}Z_{\alpha,b} \xrightarrow{\mathcal{L}} |N|.$$

Convergence is in the total variation sense. In fact, using the Stirling bounds used in our proof, we see that the total variation distance decreases to zero at the rate $O(1/b)$. $\square$

Speed-up: combining the steps

The rejection algorithm requires at least one computation of Zolotarev's function. The Zolotarev variate, in turn, is used in the generation of $T_{\alpha,\beta}$ via Theorem 1, which requires another evaluation of Zolotarev's function. Luckily, both can be combined into one, saving one expensive function evaluation, which should cut computational time almost in half: indeed, $B(x) = 1/A(x)^{1-\alpha}$. For the sake of clarity for those wishing to program our method, we give the combined algorithm.

```
set  σ = 1/√(β(1 − α)).
set  B(0) = α^(−α)(1 − α)^(−(1−α)).

if  σ ≥ √(2π):
   then repeat:  Generate U uniformly on [0, π] and V uniformly on [0, 1].
                 Set X ← U.
                 Compute W ← B(X) = sin(X) / ((sin(αX))^α (sin((1−α)X))^(1−α))
         until V ≤ (W/B(0))^(β/α)
   else repeat:  Generate N standard normal, and V uniformly on [0, 1].
                 Set X ← σ|N|.
                 Compute W ← B(X) = sin(X) / ((sin(αX))^α (sin((1−α)X))^(1−α))
         until X ≤ π and V e^(−N²/2) ≤ (W/B(0))^(β/α)
(X is now ready, but we only need W = B(X))

generate a gamma random variate G =ᴸ G_(1+β(1−α)/α).
return 1/(W G^(1−α))^(1/α)
```

The exponentially tilted unilateral stable distribution

If X is a random variable possessing density f on the positive halfline, then the exponentially tilted random variable X_λ has density proportional to $e^{-\lambda x} f(x)$, where $\lambda \geq 0$ is the tilting parameter. Random variate generation could trivially be done by rejection, and this is a common recommendation in recent research work (see, e.g., Brix (1996)):

```
repeat generate X with density f
        generate V uniformly on [0, 1]
until V ≤ e^{-λX}
return X
```

The probability that X generated in the first step is accepted is

$$\mathbf{E}\left\{e^{-\lambda X}\right\} = \mathcal{L}(\lambda),$$

where $\mathcal{L}$ is the Laplace transform of X. The expected number of iterations before halting is thus $1/\mathcal{L}(\lambda)$. As $\lambda \to \infty$, this becomes quite unacceptable. For example, if $X = S_\alpha$, then X_λ is the exponentially tilted unilateral stable random variable $S_{\alpha,\lambda}$, $\alpha \in (0, 1), \lambda \geq 0$. The expected complexity of the trivial rejection algorithm would be formidable:

$$\frac{1}{\mathbf{E}\left\{e^{-\lambda S_\alpha}\right\}} = e^{\lambda^\alpha}.$$

The task ahead of us is to develop an algorithm whose complexity is uniformly bounded over all values of $\alpha \in (0, 1)$ and $\lambda \geq 0$. The remainder of the paper develops just such a method.

The random variable $S_{\alpha,\lambda}$ has density

$$f_{\alpha,\lambda}(x) = e^{\lambda^\alpha - \lambda x} g_\alpha(x), x > 0.$$

We just want to highlight the main approaches to random variate generation in these situations. Exponential tilting, in all its simplicity, creates new challenges beyond those for $T_{\alpha,\beta}$.

The Laplace transform of $S_{\alpha,\lambda}$ is

$$\mathbf{E}\left\{e^{-\mu S_{\alpha,\lambda}}\right\} = \mathbf{E}\left\{e^{\lambda^\alpha - (\mu+\lambda) S_\alpha}\right\} = e^{\lambda^\alpha - (\mu+\lambda)^\alpha}.$$

It would be useful as a project to develop random variate generators given Laplace transforms instead of densities or distribution functions, but at present, no generic method of this kind exists. Instead, we rely once again on Zolotarev's integral respresentation and deduce (see Theorem 1) the density of S_α:

$$g_\alpha(x) = \frac{\alpha}{1-\alpha} x^{-\frac{1}{1-\alpha}} \frac{1}{\pi} \int_0^\pi A(u) e^{-A(u)/x^{\alpha/(1-\alpha)}} \, du.$$

A little bit of calculus shows that the density of

$$Y = S_{\alpha,\lambda}^{-\alpha/(1-\alpha)} \tag{3}$$

is $\int_0^\pi h(y,u)\,du$, which is just the marginal density of the bivariate density (in (y,u)) on $[0,\infty) \times [0,\pi]$:

$$h(y,u) = \frac{A(u)e^{\lambda^\alpha}}{\pi} \exp\left(-\lambda y^{-\frac{1-\alpha}{\alpha}} - A(u)y\right). \tag{4}$$

This is not a form that lends itself easily to a mixture representation. We will sample (Y,U) from h by the method called double rejection, and then return $S_{\alpha,\lambda} = Y^{-(1-\alpha)/\alpha}$. Turning the one-dimensional problem into a two-dimensional one will prove to be a good thing.

REMARK 3. The references cited in the introduction often use an additional parameter in tilting and define a family with Laplace transform (in μ)

$$e^{\theta(\lambda^\alpha - (\mu+\lambda)^\alpha)},$$

where θ is a new parameter. For the case that interests us, $\theta > 0$, this is easily seen to be the Laplace transform of

$$\theta^{\frac{1}{\alpha}} S_{\alpha,\lambda\theta^{\frac{1}{\alpha}}}.$$

Thus, θ is not a true new shape parameter. $\square$

REMARK 4. Rosinski (2007a, 2007b) discusses the simulation of exponentially and polynomially tilted random vectors, where tilting is applied to the radius of the vectors. Uniformly fast generation in the sense of our paper should prove an interesting future research project. Rosinski also discusses the simulation of Lévy, stable, gamma and Bessel processes, while allowing approximations. For example, some processes can be represented as infinite sums of functions of point processes, providing a fresh and natural angle to process simulation. However, exact simulation, even at one point in time, would require infinite resources, as we cannot truncate the sum without making an error. It is of interest to develop series representations with a random but almost surely finite number of terms. $\square$

REMARK 5. Gamma tilting. In this paper, we are only considering polynomial and exponential tilting. Gamma tilting (Barndorff-Nielsen and Shephard, 2001) is also of interest in distributional theory. Here tilting is done by a factor of the form $x^\beta e^{-\lambda x}$, with β in an appropriate range (to make the resulting function integrable) and $\lambda \geq 0$ (for quickly decreasing densities, one could also relax the condition that $\lambda \geq 0$). It is worthwhile to extend the methods of this paper to this situation as well. In fact, one might as well tackle general tilting by a factor $e^{\psi(x)}$. We note here that if X has log-concave density f, and ψ is concave, then $e^{\psi(x)}f(x)$ is log-concave. Examples include gamma tilting with $\lambda \in \mathbf{R}$ and $\beta \geq 0$, normal tilting by $e^{-\lambda x^2}$ (as proposed for the stable law by Barndorff-Nielsen and Shephard in 2001), and tilting by functions of the form $x^\beta e^{-\lambda x^\gamma}$. A first point of attack should be the universal and uniformly fast log-concave generator of Devroye (1984). However, this requires the availability, at unit cost, of the value of the density f, a condition that is not satisfied for the stable law. Therefore, gamma tilted stable random variables will pose a (modest) technical hurdle. $\square$

The double rejection method

To generate a random variate from the density

$$f(x) = \int h(x, u) \, du,$$

where h is a given bivariate density, the following method, dubbed the double rejection method, can be used. We discuss it separately from the problem at hand because it is so flexible that many applications in other situations can be foreseen.

The situations covered are the following: a marginal random variate X is to be generated when its bivariate law (X, U) has a known density. The (marginal) density of X is thus given as an integral, and exact random variate generation seems somehow difficult. One can generate X by first generating the marginal random variate U, and then by drawing X from the joint density of (X, U) conditional on U. However, the density of U too is only known as an integral, and random variate generation is difficult. The double rejection method is applicable if we can find a dominating curve $g \geq h$ with the property that its marginal integral g^* is easy to compute explicitly:

$$g^*(u) = \int g(x, u) \, dx.$$

Let $G^* = \int g^*(u) \, du$ be its integral, so that g^*/G^* is a density in u. If we are lucky and g^* is easy to sample from, and g, considered as a density in x for u fixed, is easy to sample from, then the rejection method would be as follows:

```
repeat generate U with density g*/G*
        generate X with density proportional to g(x, U)
        generate V uniformly on [0, 1]
until  Vg(X, U)/h(X, U) ≤ 1
return X
```

The expected number of iterations is easily seen to be G^*.

REMARK 6. Superficially, it may seem that taking U outside the loop accomplishes the same thing. This is indeed the case:

```
generate U with density g*/G*
repeat generate X with density proportional to g(x, U)
        generate V uniformly on [0, 1]
until  Vg(X, U)/h(X, U) ≤ 1
return X
```

However, the expected number of iterations, given U is $g^*(U)/h^*(U)$, where $h^*(u) = \int h(x,u)\,dx$. Thus, unconditioning, the expected number of iterations is $\int (g^*(u))^2/h^*(u)\,du/G^*$, and it is easy to find examples in which this is infinite. In any case, it is always more than G^*, a fact easily established by the Cauchy-Schwarz inequality. In fact, the procedure may not halt if $g^* > 0$ and $h^* = 0$ on a set of positive Lebesgue measure. A similar phenomenon was pointed out by Devroye (1986) in the context of taking the rejection variable (V above) out of the loop in the rejection algorithm. $\square$

It is easy to see that (X, U) generated at the outset of each loop has density proportional to $g(x, u)$, so that upon exit, the pair (X, U) has density $h(x, u)$, and thus, X has density f. While g^* may be easy to compute, and generation from $g(x, u)$ for u fixed may be simple to implement, it is sometimes the case, as the present example will illustrate, that g^* is not always easy to sample from. In that case, a second level of rejection is often handy. For easy reading, we will use double stars. So, there is a function $g^{**} \geq g^*$ which can be used to sample from g^* by rejection. We summarize:

```
repeat repeat generate U with density proportional to g**
              generate W uniformly on [0,1]
       until Z ≝ Wg**(U)/g*(U) ≤ 1
       (note:  U now has density g*/G*)
       generate X with density proportional to g(x,U)
       generate V uniformly on [0,1]
       (note:  we can take V = Z)
until Vg(X,U)/h(X,U) ≤ 1
return X
```

The remark "we can take $V = Z$" refers to the fact conditional on Z having been obtained as a result of the inner loop, Z is necessarily uniformly distributed on $[0, 1]$ and independent of (X, U). The expected number of outer loops is still $\int g^*$. By Wald's identity, it is easily established that the expected number of inner loops executed is $\int g^{**}$. We hope that the example below will convince the readers of the utility of this method for many distributions, especially distributions that are described via integral representations.

The R distribution

The function $g(y, u)$ we will use for fixed u takes a three-parameter bi-exponential form, which we take the liberty of calling the R distribution. We cannot progress without developing a good generator for it, and presenting its marginal law in u. The $R(\lambda, a, b)$ distribution has density

$$r(y) = ac \exp(-\lambda y^{-b} - ay), y > 0,$$

where $\lambda \geq 0$, $b > 0$ and $a > 0$ are parameters, and c is a normalization constant.

Set $L(y) = \log r(y)$, and check that $L'(y) = b\lambda/y^{b+1} - a$, $L''(y) = -b(b+1)/y^{b+2}$, and that all higher derivatives are of alternating signs. Thus, r is log-concave, with a peak at

$$m \stackrel{\text{def}}{=} \left(\frac{b\lambda}{a}\right)^{\frac{1}{b+1}}.$$

Let $\delta > 0$ be a number to be determined. The following bound follows from the properties given above:

$$L(y) \leq \log r^*(y) \stackrel{\text{def}}{=} \begin{cases} L(m) - \frac{(y-m)^2}{2}|L''(m)| & (y \leq m), \\ L(m) & (m \leq y \leq m + \delta), \\ L(m) - (y - m - \delta)|L'(m+\delta)| & (y \geq m + \delta). \end{cases}$$

The areas under the three pieces of the bounding curve are respectively, $a_1 = \int_{-\infty}^{m} r^*(y)dy$, $a_2 = \int_{m}^{m+\delta} r^*(y)dy$ and $a_3 = \int_{m+\delta}^{\infty} r^*(y)dy$. This yields $a_1 = e^{L(m)}\sqrt{\pi/(2|L''(m)|)}$, $a_2 = e^{L(m)}\delta$, and $a_3 = \frac{e^{L(m)}}{|L'(m+\delta)|}$.

Set $Q = 1 + 1/b$ and verify that

$$e^{L(m)} = ac\, e^{-amQ}, |L''(m)| = \frac{(b+1)a}{m}, |L'(m+\delta)| = a \times \frac{\left(1 + \frac{\delta}{m}\right)^{b+1} - 1}{\left(1 + \frac{\delta}{m}\right)^{b+1}}.$$

A random variate Y with density proportional to r^* is easy to generate. We should proceed as follows:

$$Y \stackrel{\mathcal{L}}{=} \begin{cases} m - \frac{|N|}{\sqrt{|L''(m)|}} & \text{with probability } \frac{a_1}{a_1+a_2+a_3}, \\ m + V\delta & \text{with probability } \frac{a_2}{a_1+a_2+a_3}, \\ m + \delta + \frac{E}{|L'(m+\delta)|} & \text{with probability } \frac{a_3}{a_1+a_2+a_3}. \end{cases}$$

Here N is standard normal, E is exponential and V is uniform on $[0, 1]$.

Finally, the rejection step is implemented by accepting Y if $V^*r^*(Y)/r(Y) \leq 1$, or equivalently, if $\log(r^*(Y)/r(Y)) \leq E^*$, where V^* is uniform $[0, 1]$ and E^* is exponential, independent of all other random variables. Note that in this test, the coefficient ac cancels out, so that we do not need to know the normalization constant.

In what follows, we only need to know

$$R^* \stackrel{\text{def}}{=} \int r^*(y)\, dy$$

$$= e^{L(m)}\left(\sqrt{\frac{\pi}{2|L''(m)|}} + \delta + \frac{1}{|L'(m+\delta)|}\right)$$

$$= ce^{-amQ}\left(\sqrt{\frac{\pi m a}{2(b+1)}} + a\delta + \frac{\left(1+\frac{\delta}{m}\right)^{b+1}}{\left(1+\frac{\delta}{m}\right)^{b+1} - 1}\right). \tag{5}$$

It is clear that δ must be chosen to minimize R^*. On the other hand, we do not wish to clutter our algorithms. We will derive an upper bound for R^*, and that bound will immediately suggest a choice for δ. Using the general bound for $x, t > 0$,

$$\frac{(1+x)^t}{(1+x)^t - 1} = \frac{1}{1 - (1+x)^{-t}} \le \frac{1}{1 - (1+tx)^{-1}} = 1 + \frac{1}{tx},$$

we note that

$$R^* \le ce^{-amQ}\left(\sqrt{\frac{\pi m a}{2(b+1)}} + a\delta + 1 + \frac{m}{(b+1)\delta}\right).$$

Now we optimize with respect to δ, equating the two terms that depend upon it: $\delta = \sqrt{m/((b+1)a)}$, to conclude that

$$R^* \le ce^{-amQ}\left(\sqrt{\frac{ma}{(b+1)}}\left(\sqrt{\frac{\pi}{2}} + 2\right) + 1\right). \tag{6}$$

The details for the exponentially tilted stable generator worked out

We return now to the formalism and notation of the double rejection method. The density of (Y, U) on $[0, \infty) \times [0, \pi]$ is $h(y, u)$, given by (4). Recalling the bound $h(y, u) \le g(y, u)$, we note that for fixed u, $g(y, u)$ is proportional to the density of a $R(\lambda, A(u), (1-\alpha)/\alpha)$ random variable. In the previous section, we should thus replace the parameters as follows, keeping λ the same: $a := A(u)$, $b := (1-\alpha)/\alpha$ (so, $b + 1 = 1/\alpha$), $c := A(u)\exp(\lambda^\alpha)/\pi$, $Q := 1/(1-\alpha)$, and $m := ((1-\alpha)\lambda/(\alpha A(u))^\alpha$. Recalling the notation $B(x)$ from Zolotarev's distribution, with $B(0) = 1/(\alpha^\alpha(1-\alpha)^{1-\alpha})$, and $A(x)$ for Zolotarev's function, we see that amQ in the previous section becomes $B(0)A(u)^{1-\alpha}\lambda^\alpha$ and $am/(b+1)$ is $B(0)A(u)^{1-\alpha}\lambda^\alpha\alpha(1-\alpha)$. Since $A(u)^{1-\alpha} = 1/B(u)$, the double rejection method would need the function $g^*(u)$ given by R^* (formula (5)). Define

$$\gamma = \lambda^\alpha \alpha(1-\alpha).$$

Then

$$g^*(u) = \frac{1}{\pi} \times e^{\lambda^\alpha\left(1 - \frac{B(0)}{B(u)}\right)}\left(\left(1 + \sqrt{\frac{\pi}{2}}\right)\sqrt{\frac{B(0)}{B(u)}}\gamma + \frac{\left(\sqrt{\gamma} + \frac{\alpha}{\sqrt{B(0)/B(u)}}\right)^{\frac{1}{\alpha}}}{\left(\sqrt{\gamma} + \frac{\alpha}{\sqrt{B(0)/B(u)}}\right)^{\frac{1}{\alpha}} - \left(\sqrt{\gamma}\right)^{\frac{1}{\alpha}}}\right).$$

This is proportional to an unwieldy density, and that is precisely why the double rejection trick is so useful. We will use $g^* \leq g^{**}$ using bound (6). With the parameters properly replaced, this yields:

$$g^*(u) \leq \frac{1}{\pi} \times e^{\lambda^\alpha \left(1 - \frac{B(0)}{B(u)}\right)} \left(\left(2 + \sqrt{\frac{\pi}{2}}\right) \sqrt{\frac{B(0)}{B(u)}} \gamma + 1\right).$$

But this is not better, because we still have the dependence upon $B(u)$. However, we recall the gaussian upper bound and linear lower bound for $B(u)$, and estimate the last expression from above by

$$\frac{1}{\pi} \times e^{\lambda^\alpha \left(1 - e^{\frac{\alpha(1-\alpha)u^2}{2}}\right)} \left(\left(2 + \sqrt{\frac{\pi}{2}}\right) \sqrt{\frac{\pi}{\pi - u}} \gamma + 1\right) \leq \frac{1}{\pi} \times e^{-\frac{\gamma u^2}{2}} \left(\left(2 + \sqrt{\frac{\pi}{2}}\right) \sqrt{\frac{\gamma \pi}{\pi - u}} + 1\right).$$

Define

$$\xi \overset{\text{def}}{=} \frac{\left(2 + \sqrt{\frac{\pi}{2}}\right) \sqrt{2\gamma} + 1}{\pi},$$

and

$$\psi = \frac{e^{-\frac{\gamma \pi^2}{8}}}{\pi} \left(2 + \sqrt{\frac{\pi}{2}}\right) \times \sqrt{\gamma \pi}.$$

Simple bounding on the intervals $[0, \pi/2]$ and $[\pi/2, \pi]$ separately shows that we have

$$g^*(u) \leq g^{**}(u) \overset{\text{def}}{=} \begin{cases} \xi e^{-\frac{\gamma u^2}{2}} 1_{[u \geq 0]} + \frac{\psi}{\sqrt{\pi - u}} 1_{[0 < u < \pi]} & \text{if } \gamma \geq 1; \\ \xi 1_{[0 \leq u \leq \pi]} + \frac{\psi}{\sqrt{\pi - u}} 1_{[0 < u < \pi]} & \text{if } \gamma < 1. \end{cases}$$

The function g^{**} is not only easy to evaluate, but it is also proportional to a density that is easily sampled from.

In the definition of g^{**}, we recognize three components, a gaussian, a beta $(1/2, 1)$, and a uniform. The areas under these three curves are, respectively,

$$w_1 = \xi \sqrt{\pi/(2\gamma)},$$
$$w_2 = 2\psi \sqrt{\pi},$$
$$w_3 = \xi \pi.$$

Sampling is thus done as a mixture, as shown below. Note that the support of g^{**} is the positive halfline, not $[0, \pi]$. This is done to avoid an evaluation of the gaussian integral, but is inconsequential, as $g^*/g^{**} = 0$ outside $[0, \pi]$.

```
(generator for a random variate with density proportional to g**)

generate V uniformly on [0, 1]
if γ ≥ 1 then if V < w₁/(w₁+w₂) then set X ← |N|/√γ where N is standard normal
                          else set X ← π(1 − W²) where W is uniform [0, 1]
        else generate W uniformly on [0, 1]
              if V < w₃/(w₃+w₂) then set X ← πW
                          else set X ← π(1 − W²)
return X
```

We conclude this section and our struggle with a complexity result.

THEOREM 3. *Let $R(\lambda, \alpha)$ denote the expected number of loops necessary to generate a random variate Y (and thus $S_{\alpha,\lambda}$) by the double rejection method outlined above. Then*

$$\sup_{\alpha \in (0,1), \lambda \geq 0} R((\lambda, \alpha) \leq \sqrt{8} + \sqrt{\pi} + 1 + \frac{8}{\pi\sqrt{e}} + \sqrt{\frac{8}{\pi e}} \approx 8.11328125.$$

PROOF. As remarked above, the expected number of iterations is equal to $\int g^{**}$, which is $w_1 + w_2$ when $\gamma \geq 1$ and $w_2 + w_3$ when $\gamma \leq 1$. First of all, this integral only depends upon the parameters through $\gamma \geq 0$. For $\gamma \leq 1$,

$$w_3 = \xi\pi \leq \sqrt{8} + \sqrt{\pi} + 1.$$

Using the fact that $\sqrt{t}e^{-t} \leq 1/\sqrt{2e}$, we see that

$$\sup_{\gamma \geq 0} w_2 \leq \frac{8}{\pi\sqrt{e}} + \sqrt{\frac{8}{\pi e}}.$$

Finally, for $\gamma \geq 1$,

$$w_1 = \xi\sqrt{\frac{\pi}{2\gamma}} \leq \sqrt{\frac{1}{2\gamma\pi}} + \frac{2}{\sqrt{\pi}} + \frac{1}{\sqrt{2}} \leq \sqrt{\frac{1}{2\pi}} + \frac{2}{\sqrt{\pi}} + \frac{1}{\sqrt{2}},$$

which is less than the bound on w_1. Therefore,

$$\sup_{\gamma \geq 0} \int g^{**} \leq \sqrt{8} + \sqrt{\pi} + 1 + \frac{8}{\pi\sqrt{e}} + \sqrt{\frac{8}{\pi e}}. \quad \square$$

I am sure that the uniform bound in Theorem 3 can be somewhat improved.

Appendix: the full algorithm for exponentially tilted stable distributions

The following algorithm combines all the ideas explained above. Speed is sacrificed in the code for readability. The two rejection blocks can clearly be distinguished within the algorithm. The ratio $B(x)/B(0)$ referred to in the algorithm is defined in the section on Zolotarev's distribution. For numeric stability it should be implemented using the sinc function as suggested there.

It is also noteworthy that in the case of no tilting ($\lambda = 0$), a trace of the algorithm reveals that in the first loop, U is uniform on $[0, \pi]$, and that $X = E'/A(U)$ (in the notation of the algorithm below). We exit with $(A(U)/E')^{(1-\alpha)/\alpha}$ without looping. Thus, the algorithm reduces to Kanter's method. Since the design is continuous in the parameters, we return for small values of λ a random variate close to that most of the time, and the acceptance ratio should be near one.

`set-up:` `define` $\gamma = \lambda^\alpha \alpha(1-\alpha)$, $\xi = \pi^{-1}\left(2+\sqrt{\frac{\pi}{2}}\right)\sqrt{2\gamma}+1$, $\psi = \pi^{-1}e^{-\frac{\gamma\pi^2}{8}}\left(2+\sqrt{\pi/2}\right)\times\sqrt{\gamma\pi}$,

$\qquad\qquad$ $w_1 = \xi\sqrt{\pi/(2\gamma)}$, $w_2 = 2\psi\sqrt{\pi}$, $w_3 = \xi\pi$, $b = (1-\alpha)/\alpha$

`repeat repeat generate` U `with density proportional to` g^{**}:

$\qquad\qquad$ `generate` V, W' `uniformly on` $[0,1]$

$\qquad\qquad$ `if` $\gamma \geq 1$ `then if` $V < \frac{w_1}{w_1+w_2}$ `then` $U \leftarrow |N|/\sqrt{\gamma}$ `where` N `is standard normal`

$\qquad\qquad\qquad\qquad$ `else` $U \leftarrow \pi(1-W'^2)$

$\qquad\qquad$ `else if` $V < \frac{w_3}{w_3+w_2}$ `then` $U \leftarrow \pi W'$

$\qquad\qquad\qquad\qquad$ `else` $U \leftarrow \pi(1-W'^2)$

$\qquad$ `(note:` U `has density proportional to` g^{**}`)`

$\qquad$ `generate` W `uniformly on` $[0,1]$

$\qquad$ $\zeta = \sqrt{B(U)/B(0)}$, $\phi = \left(\sqrt{\gamma}+\alpha\zeta\right)^{\frac{1}{\alpha}}$, $z = \phi/\left(\phi-\left(\sqrt{\gamma}\right)^{\frac{1}{\alpha}}\right)$

$\qquad$ $\rho = \dfrac{\pi\times e^{-\lambda^\alpha\left(1-\zeta^{-2}\right)}\times\left(\xi e^{-\frac{\gamma U^2}{2}}1_{[U\geq 0,\gamma\geq 1]}+\frac{\psi}{\sqrt{\pi-U}}1_{[0<U<\pi]}+\xi 1_{[0\leq U\leq\pi,\gamma<1]}\right)}{\left(1+\sqrt{\frac{\pi}{2}}\right)\frac{\sqrt{\gamma}}{\zeta}+z}$

$\qquad$ `until` $U < \pi$ `and` $Z \stackrel{\text{def}}{=} W\rho \leq 1$ `(note:` $\rho = g^{**}(U)/g^*(U)$`)`

$\qquad$ `(note:` U `now has density` g^*/G^* `and` Z `is uniform` $[0,1]$`)`

$\qquad$ `generate` X `with density proportional to` $g(x,U)$

$\qquad\qquad$ `set-up of constants:` $a = A(U)$, $m = (b\lambda/a)^\alpha$, $\delta = \sqrt{m\alpha/a}$,

$\qquad\qquad\qquad\qquad$ $a_1 = \delta\sqrt{\pi/2}$, $a_2 = \delta$, $a_3 = z/a$, $s = a_1 + a_2 + a_3$

$\qquad\qquad$ `generate` V' `uniform on` $[0,1]$

$\qquad\qquad$ `if` $V' < a_1/s$ `then generate` N' `standard normal`

$\qquad\qquad\qquad\qquad$ $X \leftarrow m - \delta|N'|$

$\qquad\qquad$ `else if` $V' < a_2/s$ `then` $X \leftarrow$ `uniform on` $[m, m+\delta]$

$\qquad\qquad\qquad\qquad$ `else generate` E' `exponential`

$\qquad\qquad\qquad\qquad$ $X \leftarrow m + \delta + E'a_3$

$\qquad$ `(note:` X `has density proportional to` $g(x,U)$`)`

$\qquad$ `generate` E `exponential (note: we can take` $E = -\log Z$`)`

`until` $X \geq 0$ `and` $a(X-m) + \lambda(X^{-b} - m^{-b}) - \frac{N'^2}{2}1_{[X<m]} - E'1_{[X>m+\delta]} \leq E$

`return` $1/X^b$

Acknowledgments

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