SIMULATING PERPETUITIES

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Abstract. A perpetuity is a random variable that can be represented as $1 + W_1 + W_1W_2 + W_1W_2W_3 + \cdots$, where the W_i's are i.i.d. random variables. We study exact random variate generation for perpetuities and discuss the expected complexity. For the Vervaat family, in which $W_1 \stackrel{\mathcal{L}}{=} U^{1/\beta}$, $\beta > 0$, U uniform $[0,1]$, all the details of a novel rejection method are worked out. There exists an implementation of our algorithm that only uses uniform random numbers, additions, multiplications and comparisons.

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Introduction

Following Embrechts and Goldie (1994), a perpetuity is a random variable

$$Y = 1 + W_1 + W_1W_2 + W_1W_2W_3 + \cdots + W_1W_2 \ldots W_n + \cdots$$

where the W_i's are i.i.d. random variables distributed as W and $\mathbb{E}W = m < 1$. It occurs in various fields such as financial modelling, hydrology, insurance mathematics and the analysis of Hoare's selection algorithm. Key references are Takács (1955), Kesten (1973), Vervaat (1979), Goldie and Grübel (1995), Grübel and Rösler (1996) and de Bruijn (1951). These are special cases of more general random variables that may be written as

$$A_0 + A_1W_1 + A_2W_1W_2 + A_3W_1W_2W_3 + \cdots$$

(Embrechts, Klüppelberg and Mikosch, 1997, section 8.4) and that occur as solutions of random recurrence relations or in financial mathematics. The question we ask here is how we can exactly generate such random variates given that we have an infinite source of i.i.d. uniform $[0, 1]$ random variates at our disposal. Approximations by appropriately truncating tails are not allowed. We propose a solution based upon the rejection method. The density of Y is never needed however, as convergent approximations suffice to make correct acceptance/rejection decisions. The task at hand is quite formidable: it requires the development of good upper bounds and explicit approximations for the density of Y. We will define the general strategy, and then work out the details when $W = U^{1/\beta}$, $\beta > 0$, where U is uniform $[0, 1]$. We are aware of other attempts to solve this problem. Approximate random variate generation for Y in this situation was considered by Chamayou (1997). As our perpetuities form a special subfamily of the infinitely divisible distributions, one might consider applying general approximation algorithms for infinitely divisible distributions with given Lévy measures. The earliest mention of this is in Bondesson (1982) and Bouleau (1988). More recent work by Damien, Laud and Smith (1995), and Walker (1996) uses Gibbs sampling. The present paper demonstrates that it is possible to generate perpetuities exactly in finite time, but it does not claim that these methods are inherently practical. Indeed, various design constants and inequalities are deliberately picked in suboptimal ways to keep the paper relatively simple. After we described the problem (with $\beta = 1$) to Jim Fill, he found an algorithm based on the perfect simulation paradigm proposed by Propp and Wilson (1996). We have not seen it to date. Our approach has the advantage of showing a worked out example that may be used as a prototype for many infinitely divisible distributions, even those that are not perpetuities.

To keep the notation consistent, we use to following symbols throughout the paper: W is the basic random variable in the definition of a perpetuity Y. The random variables $W_1, W_2, \ldots$ are i.i.d. and distributed as W. Sometimes, it is convenient to work with $Z = Y - 1$. Let $\mu_k \stackrel{\text{def}}{=} \mathbb{E}\{W^k\}$, $m = \mu_1$, and $\sigma^2 = \mathbb{V}\{W\}$. Throughout the paper, U denotes a uniform $[0, 1]$ random variable and f is the density of Z. We say that Z is a Vervaat random variable or Vervaat perpetuity when $W = U^{1/\beta}$ for $\beta > 0$. In that case, the density of Z^β will be denoted by g.

Throughout, we assume that real numbers can be stored with infinite precision; that all standard arithmetic operations take one unit of time; and that we have a source capable of generating an infinite sequence of iid uniform $[0, 1]$ random variates $U_1, U_2, \ldots$. Some of these conditions are controversial (e.g., which operations are "standard" ?), but they are consistent with those found in many recent articles and books on the subject (see e.g. Devroye (1986b)).

Basic properties of perpetuities.

In this section, some general properties are recalled for perpetuities.

LEMMA M1. *Assume $W \ge 0$ and $m < 1$. Then*

$$Y_n \stackrel{\text{def}}{=} 1 + W_1 + W_1W_2 + W_1W_2W_3 + \cdots + W_1W_2 \dots W_n$$

tends almost surely to a limit Y, which is a random variable with mean $1/(1-m)$. The limit random variable Y satisfies the distributional identity

$$Y \stackrel{\mathcal{L}}{=} YW + 1 \ ,$$

where Y and W are independent. Also, $\mathbb{V}\{Y\} = \frac{\sigma^2}{(1-m)^2(1-m^2-\sigma^2)}$ provided that $m^2 + \sigma^2 < 1$.

PROOF. As Y_n is nondecreasing, it has a limit Y, which is possibly infinite with positive probability. However, by monotone convergence, $\mathbb{E}Y = \lim_{n\to\infty} \mathbb{E}Y_n = \lim_{n\to\infty} \sum_{i=0}^n m^i = 1/(1-m) < \infty$. Thus, $Y < \infty$ almost surely. □

Note that if $m \ge 1$, Y_n may not tend to a proper random variable (just take $W \equiv 1$). To avoid difficulties, we will assume $m < 1$ and leave the more challenging case $m = 1$ for another time. It is easy to see that $Z \stackrel{\mathcal{L}}{=} (Z+1)W$ where Z and W are independent. The following Lemma captures what is known about the tail of Z.

LEMMA M2 (GOLDIE AND GRÜBEL, 1996).

- *A. If $0 \le W \le 1$ and $\mathbb{P}\{W = 1\} < 1$, then $\mathbb{P}\{Z > t\} \le e^{t(\log m + o(1))}$.*
- *B. If $0 \le W$ and $\mathbb{P}\{W > 1\} > 0$, then $\mathbb{P}\{Z > t\} \ge t^{c+o(1)}$ for some $c > -\infty$.*
- *C. If $0 \le W \le 1$ and W has a density f which remains bounded away from 0 and 1 on $[1-\epsilon, 1]$ for some small $\epsilon > 0$, then $\mathbb{P}\{Z > t\} \le e^{-(1+o(1))t \log t}$.*

It is useful to have recurrences for all the moments of Y and Z. Clearly,

$$\mathbb{E}\{Z^k\} = \mu_k \mathbb{E}\{(Z+1)^k\}$$

from which we easily derive the recurrences

$$\mathbb{E}\{Z^k\} = \frac{\mu_k}{1-\mu_k} \times \sum_{j=0}^{k-1} \binom{k}{j} \mathbb{E}\{Z^j\} \ .$$

In particular, $\mathbb{E}\{Z\} = m/(1-m)$ and $\mathbb{E}\{Z^2\} = (\mu_2/(1-\mu_2))(1 + 2m/(1-m))$. Note that

$$\mathbb{E}\{Y^k\} = \sum_{j=0}^{k} \binom{k}{j} \mathbb{E}\{Y^j\}\mathbb{E}\{W^j\} \ge \mathbb{E}\{Y^k\}\mathbb{E}\{W^k\}$$

so that for all moments of Y to be finite, it is necessary that all moments of W be less than one, and thus $|W| \leq 1$ almost surely. We will assume $0 \leq W \leq 1$.

REMARK: GENERAL BOUNDS ON THE DENSITY. Useful bounds on the density f are easily derived when W has a nonincreasing density. A perpetuity is said to be monotone if W has a nonincreasing density on $[0, \infty)$ and if $Z < \infty$ almost surely (so that Z is a proper random variable). A monotone perpetuity Z clearly has a nonincreasing density. Furthermore, as $\mathbb{P}\{Z < t\} \leq \mathbb{P}\{W < t\}$, we see that $f(0) \leq f_W(0)$, where f_W is the density of W. From all the above, we may deduce simple upper bounds on the density f in terms of f_W. For example, by Devroye (1986b, p. 313),

$$f(z) \leq \min\left(f_W(0), \frac{(r+1)\mathbb{E}\{Z^r\}}{z^{r+1}}\right) ,$$

where $r > 0$. With $r = 1$, we thus have

$$f(z) \leq \min\left(f_W(0), \frac{2m}{(1-m)z^2}\right) .$$

This inequality may thus be used as the basis of a rejection method because

$$\sqrt{\frac{2m}{f_W(0)(1-m)}} \times \frac{U_1}{U_2}$$

has a density proportional to the upper bound when U_1, U_2 are independent uniform $[0, 1]$ random variables (Devroye (1986b, p. 315)).

REMARK: GIBBS SAMPLER. Based upon the distributional identity $Z \stackrel{\mathcal{L}}{=} W(Z+1)$, a Gibbs sampler may be constructed, but we do not accept that as it only yields an approximative solution. It could be based upon the following algorithm, suitably stopped:

```
Z ← c
repeat forever:  generate W
                 Z ← W(Z + 1)
```

If this algorithm is stopped after n iterations, the value of Z at that time is distributed as

$$W_1 + W_1W_2 + \cdots + W_1W_2\cdots W_{n-1} + cW_1W_2\cdots W_{n-1} \ .$$

The monotonicity argument of Lemma M1 shows that this converges in distribution to Z for any value of c. It does not imply convergence of densities. If f_n is the density of Z after the n-th iteration, then we denote by Tf_n the density of $W(Z+1)$. Thus, as $Tf = f$,

$$\int |f_{n+1} - f| = \int |Tf_n - Tf| \le \int |f_n - f| \ ,$$

so the total variation distance decreases monotonically. The rate of decrease to zero is governed by the distribution of W.

The characteristic function.

In special cases, the characteristic function of Z is easy to derive. From the recurrence, it is easy to show (see Vervaat, 1979) that if $W \stackrel{\mathcal{L}}{=} U^{1/\beta}$ where U is uniform $[0,1]$, then Z has characteristic function

$$\phi(t) = \exp\left(\beta \int_0^1 \frac{e^{itx} - 1}{x}\, dx\right) = \exp\left(\beta \int_0^t \frac{e^{is} - 1}{s}\, ds\right) \ . \tag{1}$$

See also de Bruijn (1951) and Takács (1954, 1955), who obtained ϕ as a limit characteristic function in a certain random process. Interestingly, for $\beta = 1$, this is also the limit distribution of $(1/n)\sum_{i=1}^n Y_i$, where the Y_i's are independent and $Y_i = i$ with probability $1/i$ and 0 otherwise. However, the latter fact is of no use in the design of an exact method. If e^{itx} here is replaced by $\psi(tx)$ for a characteristic function ψ that corresponds to a finite mean random variable, then ϕ is still a valid characteristic function (Takács, 1954). The family of distributions above is called the Takács family for general ψ and the Vervaat family with shape parameter β for (1). If W is distributed as $-U^{1/\beta}$ however, then $-Z$ is beta $(\beta, \beta+1)$ (Chamayou and Schorr, 1975). Further examples and a more thorough study of perpetuities in general can be found in the work of Dufresne (1990, 1996, 1998).

Finally, we note that (1) is in the Lévy form for infinitely divisible distributions, with Lévy measure given by $(\beta/x)\mathbb{1}_{[0,1]}(x)dx$. What will be said below can, in many cases, be extended to other infinitely divisible distributions with suitable changes in the various approximations and bounds.

The rejection method

In this paper, we find an approximation of $f(x)$ denoted by $f_n(x)$, and explicit bounds $R_n(x)$ such that

$$|f(x) - f_n(x)| \le R_n(x)$$

and $\lim_{n\to\infty} R_n(x) = 0$. We also find an explicit upper bound $h(x)$ for $f(x)$ with $\int_0^\infty h = c < \infty$ such that the following rejection method is applicable:

```
repeat
  generate U uniform [0,1]
  generate X with density proportional to h
  set T ← Uh(X)
  Accept ← [T < f(X)]
until Accept
return X
```

The decision "Accept $\leftarrow [T < f(X)]$" in the algorithm above is carried out as follows:

```
n ← 1
repeat forever:
  if T < f_n(X) − R_n(X) then return ``Accept = true''
  if T > f_n(X) + R_n(X) then return ``Accept = false''
  n ← n + 1
```

If for all x, $R_n(x) \to 0$, and $|f - f_n| \leq R_n$, then the last little piece of code correctly sets the value of the Boolean variable "Accept". It does so in a random but finite time, with probability one. This approximation trick combined with the rejection method was developed by the author in two papers in 1981, where an exact generator for the Kolmogorov-Smirnov distribution was derived. For a formal treatment, see Devroye (1986b, pp. 151–171). This method was suggested for harder problems, such as when only characteristic functions are explicitly known (Devroye, 1986a), when only moments or Fourier coefficients are known (Devroye, 1989), or when a generating function is given (Devroye, 1991).

With this algorithm design methodology, we need only find a suitable upper bound h for f, and a suitable pair (f_n, R_n). This is done in the remainder of the paper for the Vervaat perpetuities.

Upper bounds for the Vervaat family

LEMMA V1. *The density f of Z satisfies*

$$f(z) \leq h(z) \stackrel{\text{def}}{=} \min\left(\frac{(\beta+1)\beta}{z^2}, \beta z^{\beta-1}\right) .$$

PROOF. Recall

$$\mathbb{E}\{Z^k\} = \frac{\mu_k}{1-\mu_k} \times \sum_{j=0}^{k-1} \binom{k}{j} \mathbb{E}\{Z^j\} \ .$$

As $\mu_k = \mathbb{E}U^{k/\beta} = \beta/(\beta+k)$, and $\mu_k/(1-\mu_k) = \beta/k$, we see that $\mathbb{E}Z = \beta$ and $\mathbb{E}Z^2 = \frac{\beta}{2}(1+2\beta)$. Observe that $Z^\beta \stackrel{\mathcal{L}}{=} U(Z+1)^\beta$, so that Z^β has a monotonically decreasing density. Now, if g is the density of Z^β, then

$$g(0) \le \sup_{x>0} \frac{\mathbb{P}\{Z^\beta < x\}}{x} \le \sup_{x>0} \frac{\mathbb{P}\{U < x\}}{x} = 1 \ .$$

But then, if f is the density of Z,

$$f(z) = \beta z^{\beta-1} g(z^\beta) \le \beta z^{\beta-1} \ .$$

Furthermore, by an inequality of Devroye (1986b),

$$g(y) \le \frac{(r+1)\mathbb{E}\{Z^{\beta r}\}}{y^{r+1}}$$

for $r > 0$. The density f of Z then satisfies the following inequality:

$$f(z) = \beta z^{\beta-1} g(z^\beta) \le \frac{\beta(r+1) z^{\beta-1} \mathbb{E}\{Z^{\beta r}\}}{z^{\beta(r+1)}} = \frac{\beta(r+1)\mathbb{E}\{Z^{\beta r}\}}{z^{\beta r+1}} \ .$$

As only the first few integral moments of Z are easy to calculate, it is convenient to set $r = 1/\beta$. In that case, we obtain

$$f(z) \le \frac{(\beta+1)\mathbb{E}\{Z\}}{z^2} = \frac{(\beta+1)\beta}{z^2} \ .$$

Now, combine both bounds. □

The following lemma is adapted from Devroye (1986b) and permits us to generate a random variate X with density proportional to h.

LEMMA V2. *For $A, B, a, b > 0$, we have*

$$\int_0^\infty \min\left(Au^{a-1}, B/u^{b+1}\right) \, du = \frac{a+b}{ab} \left(A^b B^a\right)^{\frac{1}{a+b}} \ .$$

If $p, q > 0$ and if U, V are i.i.d. uniform $[0,1]$, then the density of $X = U^p/V^q$ is

$$\frac{1}{p+q} \min\left(x^{(1/p)-1}, 1/x^{(1/q)+1}\right) \ .$$

The density of $X' \stackrel{\text{def}}{=} (B/A)^{pq/(p+q)} X$ is

$$\frac{1}{(p+q)(B^q A^p)^{\frac{1}{p+q}}} \min\left(Ax^{(1/p)-1}, B/x^{(1/q)+1}\right) \ .$$

If we apply lemma V2 with $A = \beta$, $B = \beta(\beta+1)$, $a = \beta$, $b = 1$, then we see that

$$\int_0^\infty h = \frac{\beta+1}{\beta} \left(\beta\beta^\beta(\beta+1)^\beta\right)^{\frac{1}{\beta+1}} = (\beta+1)^{\frac{1+2\beta}{1+\beta}}$$

which is finite but grows unbounded as $\beta \to \infty$. The following random variable has density proportional to h:

$$X = (\beta+1)^{\frac{1}{\beta+1}} \frac{U^{\frac{1}{\beta}}}{V} \ .$$

Here, U and V are independent uniform $[0,1]$ random variables. Improved exponential tails and corresponding generators are presented in a later section.

Approximation bounds for the characteristic function

We define, for $t > 0$,

$$\psi(t) = \int_0^t \frac{e^{is}-1}{s} \, ds$$

and note that this may be written as a function of the sine integral function $\mathcal{S}(t) = \int_0^t \sin s/s \, ds$ and the entire cosine integral function $\mathcal{C}(t) = \int_0^t (1-\cos s)/s \, ds$:

$$\psi(t) = i\mathcal{S}(t) - \mathcal{C}(t) \ .$$

We will first describe bounds assuming that both functions are available to the user (see Abramowitz and Stegun, 1975; Spanier and Oldham, 1987, for numerical methods). We also need the fact that

$$\mathcal{C}(t) \geq \max(0, \gamma + \log t)$$

where $\gamma = 0.0772156649\ldots$ is 0.5 less than Euler's constant (Spanier and Oldham, 1987, p. 363). Recall that $\phi = e^{\beta\psi}$ is the characteristic function of the Vervaat perpetuity Z. To define f_n, we approximate the inversion formula

$$f(x) = \frac{1}{2\pi} \int e^{-itx} \phi(t) \, dt$$

by cutting off the tails, and using a simple trapezoidal rule on a compact interval. Using the fact that $\Re\{\phi(t)\} = \Re\{\phi(-t)\}$ and $\Im\{\phi(t)\} = -\Im\{\phi(-t)\}$, we observe that

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \left(\Re\{\phi(t)\} \cos(tx) + \Im\{\phi(t)\} \sin(tx) \right) \, dt \\ &= \frac{1}{\pi} \int_0^\infty e^{-\beta \mathcal{C}(t)} \left(\cos(\beta \mathcal{S}(t)) \cos(tx) + \sin(\beta \mathcal{S}(t)) \sin(tx) \right) \, dt \\ &= \frac{1}{\pi} \int_0^\infty e^{-\beta \mathcal{C}(t)} \cos(\beta \mathcal{S}(t) - tx) \, dt \ . \end{aligned}$$

We introduce two positive integers whose values will increase with n to ∞, M and L. The value of ψ (and thus, ϕ) is computed for all values j/L such that $0 \leq j/L \leq M$. This implies $ML+1$ computations. In particular, the tail contribution is simply bounded as follows:

$$\frac{1}{\pi} \left| \int_M^\infty e^{-\beta \mathcal{C}(t)} \cos(\beta \mathcal{S}(t) - tx) \, dt \right| \leq \frac{1}{\pi} \int_M^\infty t^{-\beta} \, dt = \frac{M^{1-\beta}}{\pi(\beta-1)}$$

for $\beta > 1$. For the whole range of β values, a bit more work yields the following bound:

Lemma B1. *For all positive* M,

$$\frac{1}{\pi}\left|\int_M^\infty e^{-\beta\mathcal{C}(t)}\ \cos(\beta\mathcal{S}(t)-tx)\,dt\right| \le \frac{8}{\pi x M^\beta}\ .$$

Proof. It is convenient to use the representation

$$\begin{aligned}&\frac{1}{\pi}\left|\int_M^\infty e^{-\beta\mathcal{C}(t)}\ \cos(\beta\mathcal{S}(t)-tx)\,dt\right|\\ &\quad=\frac{1}{\pi}\left|\int_M^\infty e^{-\beta\mathcal{C}(t)}\ \left(\cos(\beta\mathcal{S}(t))\cos(tx)+\sin(\beta\mathcal{S}(t))\sin(tx)\right)\ dt\right|\ .\end{aligned}$$

By partial integration,

$$\begin{aligned}\int_M^\infty e^{-\beta\mathcal{C}(t)}\ \cos(\beta\mathcal{S}(t))\cos(tx)\,dt &= -e^{-\beta\mathcal{C}(M)}\ \cos(\beta\mathcal{S}(M))\frac{\sin(Mx)}{x}\\ &\quad+\beta\int_M^\infty \left(\mathcal{C}'(t)\cos(\beta\mathcal{S}(t))+\mathcal{S}'(t)\sin(\beta\mathcal{S}(t))\right)e^{-\beta\mathcal{C}(t)}\frac{\sin(tx)}{x}\,dt\end{aligned}$$

and thus, since $\mathcal{C}'(t)=(1-\cos t)/t\in[0,2/t]$ and $\mathcal{S}'(t)=\sin t/t\in[-1/t,1/t]$, and $\mathcal{C}(t)\ge\log t$,

$$\begin{aligned}&\left|\int_M^\infty e^{-\beta\mathcal{C}(t)}\ \cos(\beta\mathcal{S}(t))\cos(tx)\,dt\right|\\ &\quad\le\frac{e^{-\beta\mathcal{C}(M)}}{x}+\beta\int_M^\infty\frac{3}{tx}e^{-\beta\mathcal{C}(t)}\,dt\\ &\quad\le\frac{1}{xM^\beta}+\beta\int_M^\infty\frac{3}{t^{1+\beta}x}\,dt\\ &\quad=\frac{4}{xM^\beta}\ .\end{aligned}$$

The term involving $\sin(\beta\mathcal{S}(t))\sin(tx)$ is bounded by the same expression, so that, multiplying with $1/\pi$, we obtain the bound. □

We set

$$f_n(x)=\frac{1}{\pi L}\sum_{j=0}^{ML-1} e^{-\beta\mathcal{C}(j/L)}\ \cos(\beta\mathcal{S}(j/L)-(j/L)x)\ . \tag{2}$$

Lemma B2. *For* M, L *positive integers, the above approximation satisfies, for all* $x>0$,

$$|f_n(x)-f(x)|\le R_n(x)\stackrel{\text{def}}{=}\frac{2M\beta+Mx}{\pi L}+\frac{8}{\pi x M^\beta}\ . \tag{3}$$

PROOF. Clearly,

$$|f_n(x) - f(x)| \leq \frac{M}{\pi} \sup_{0\leq t\leq s\leq M, |t-s|\leq 1/L} \left| e^{-\beta\mathcal{C}(s)} \cos(\beta\mathcal{S}(s) - sx) - e^{-\beta\mathcal{C}(t)} \cos(\beta\mathcal{S}(t) - tx) \right| + \frac{1}{\pi} \left| \int_M^\infty e^{-\beta\mathcal{C}(t)} \cos(\beta\mathcal{S}(t) - tx)\, dt \right| .$$

Using the triangle inequality, $|\cos u - \cos v| \leq |u - v|$, and $|e^{-u} - e^{-v}| \leq |u - v|$ $(u, v \geq 0)$, and lemma B1, the first term in the upper bound may be further bounded by

$$\begin{aligned}
&\frac{M}{\pi} \sup_{0\leq t\leq s\leq M, |t-s|\leq 1/L} \left(\beta|\mathcal{C}(s) - \mathcal{C}(t)| + e^{-\beta\mathcal{C}(t)} |\beta\mathcal{S}(s) - sx - \beta\mathcal{S}(t) + tx| \right) \\
&\leq \frac{M\beta}{\pi} \sup_{0\leq t\leq s\leq M, |t-s|\leq 1/L} (|\mathcal{C}(s) - \mathcal{C}(t)| + |\mathcal{S}(s) - \mathcal{S}(t)| + x|s-t|/\beta) \\
&\leq \frac{M\beta}{\pi L} \sup_{0\leq t\leq M} (|\mathcal{C}'(t)| + |\mathcal{S}'(t)| + x/\beta) \\
&\leq \frac{M\beta}{\pi L} \left(\sup_{0\leq t} \frac{|1 - \cos t|}{t} + \sup_{0\leq t} \frac{|\sin t|}{t} + x/\beta \right) \\
&\leq \frac{2M\beta + Mx}{\pi L} .
\end{aligned}$$

Now conclude by applying the bound of lemma B1 to the tail integral in the upper bound. □

The full algorithm

We use h, f_n and R_n from the previous two sections, but still need to offer choices for the free parameters M and L as a function of n. One possible strategy picks these parameters such that $R_n < h(x)/2^n$. A sufficient choice, by lemma B2, is

$$M = \left\lceil \left(\frac{8 \cdot 2^{n+1}}{\pi x h(x)} \right)^{\frac{1}{\beta}} \right\rceil \tag{4}$$

and

$$L = \left\lceil \frac{(2M\beta + Mx)2^{n+1}}{\pi h(x)} \right\rceil . \tag{5}$$

With such a choice for $R_n(x)$, given x, the probability of $[|T - f_n(x)| \leq R_n(x)]$ is not more than $2R_n(x)/h(x) < 2^{1-n}$, uniformly over all x. This will surely imply that the expected number of iterations (i.e., different n used until the condition is violated) is not more than 4 for any x. With these choices, we summarize the algorithm:

```
repeat
  generate U uniform [0,1]
  generate X with density proportional to h
    (generate V_1, V_2 independent uniform [0,1] random variables)
    (X ← (β+1)^{1/(β+1)} V_1^{1/β} / V_2 )
  set T ← U h(X)
    (recall definition from lemma V1:  h(X) = β min((β+1)/X^2, X^{β-1}) )
  n ← 1
  repeat
    compute M and L as in (4) and (5):
      M = ⌈ ( 8·2^{n+1} / (π X h(X)) )^{1/β} ⌉,  L = ⌈ (2Mβ + MX) 2^{n+1} / (π h(X)) ⌉
    compute f_n(X) as in (2):
      f_n(x) = (1/(πL)) Σ_{j=0}^{ML-1} e^{-βC(j/L)} cos(βS(j/L) - (j/L)x)
    compute R_n(X) as in (3):  R_n(X) = (2Mβ + MX)/(πL) + 8/(πXM^β)
    n ← n+1
  until |T - f_n(X)| ≥ R_n(X)
  Accept = [T ≤ f_n(X) - R_n(X)]
until Accept
return X
```

Complexity.

We know that the above algorithm halts with probability one, and returns a correctly distributed random variate. The expected number of outer loop iterations is $\int h < \infty$, but this is hardly an appropriate measure of the complexity of the algorithm. In the n-th iteration of the inner loop, we expend effort proportional to LM, which in turn is proportional to $2^{n(1+1/\beta)}$. The probability of not halting after n iterations decreases as 2^{1-n} for a given candidate random variable X. It should thus be clear that for any particular X, the expected work in the inner loop is infinite. For this reason, serious improvements are needed. These are sketched below.

Refinement 1: Improved complexity

The rectangular approximation of an integral used in the definition of f_n may be replaced by higher order Newton-Cotes integration formulas, such as the trapezoidal rule, Simpson's rule, and Boole's rule (see Davis and Rabinowitz, 1975). This improvement has been suggested in Devroye (1986b, p. 701) precisely for the situation at hand. Theorem 14.3.2 there covers error bounds. For example, if we use Simpson's rule in the definition of f_n, i.e.,

$$f_n(x) = \frac{1}{\pi L} \sum_{j=0}^{ML-1} \left(\frac{1}{6}\tau\left(\frac{j}{L}\right) + \frac{4}{6}\tau\left(\frac{j+1/2}{L}\right) + \frac{1}{6}\tau\left(\frac{j+1}{L}\right) \right)$$

where

$$\tau(t) \stackrel{\text{def}}{=} e^{-\beta\mathcal{C}(t)} \cos(\beta\mathcal{S}(t) - tx) \ ,$$

while maintaining the parameters M and L, then we can obtain

$$R_n(x) = \frac{\sigma M(x+m)^4}{\pi L^4} + \frac{8}{\pi x M^\beta}$$

where $\sigma \stackrel{\text{def}}{=} 1/(360\pi)$, and $m = (\mathbb{E}\{Z^4\})^{1/4}$. Recall that m is known from remarks made earlier.

THEOREM C1. *Consider the algorithm of the previous section with Simpson's rule in f_n, and with*

$$R_n(x) = \frac{\sigma M(x+m)^4}{\pi L^4} + \frac{8}{\pi x M^\beta} .$$

Furthermore, choose M as in (4) and pick

$$L = \left\lceil \left(\frac{\sigma M(x+m)^4 2^{n+1}}{\pi h(x)} \right)^{\frac{1}{4}} \right\rceil .$$

Let C be the time taken by the algorithm to process a candidate random variable X (regardless of acceptance or rejection), where each basic computation or standard function takes one time unit. Then for each $X = x$, $C < \infty$ with probability one, and if $\beta > 5/3$, we have $\mathbb{E}\{C|X = x\} < \infty$.

PROOF. From the fact that for each x, $R_n(x) \to 0$, we conclude that $C < \infty$ with probability one. Given $X = x$, the $n+1$-st iteration is reached with probability not exceeding $2R_n(x)/h(x) < 1/2^n$ by our choice of M and L. Let us write M_n and L_n to make the dependence on n explicit. Thus, the expected work is not more than a constant times

$$1 + \sum_{n=1}^{\infty} \frac{M_n L_n}{2^n} .$$

Note that

$$\begin{aligned} M_n L_n &\le \left(1 + \left(\frac{8 \cdot 2^{n+1}}{\pi x h(x)}\right)^{\frac{1}{\beta}}\right) \left(1 + \left(\frac{\sigma M(x+m)^4 2^{n+1}}{\pi h(x)}\right)^{\frac{1}{4}}\right) \\ &\le 1 + \left(\frac{8 \cdot 2^{n+1}}{\pi x h(x)}\right)^{\frac{1}{\beta}} \\ &\quad + \left(\frac{8 \cdot 2^{n+1}}{\pi x h(x)}\right)^{\frac{1}{4\beta}} \left(\frac{\sigma (x+m)^4 2^{n+1}}{\pi h(x)}\right)^{\frac{1}{4}} \\ &\quad + 2^{\left(\frac{5}{4\beta}+\frac{1}{4}\right)(n+1)} \left(\frac{8}{\pi x h(x)}\right)^{\frac{5}{4\beta}} \left(\frac{\sigma (x+m)^4}{\pi h(x)}\right)^{\frac{1}{4}} \\ &= 1 + I + II + III . \end{aligned}$$

Note that $\sum_{n=1}^{\infty} 1/2^n = 1$, $\sum_{n=1}^{\infty} I/2^n < c_1/(xh(x))^{1/\beta}$ (for $\beta > 1$),

$$\sum_{n=1}^{\infty} II/2^n < c_2(x+m)/h(x)^{1/4}(xh(x))^{1/(4\beta)}$$

(for $\beta > 1/3$), and

$$\sum_{n=1}^{\infty} III/2^n < c_3(x+m)/h(x)^{1/4}(xh(x))^{5/(4\beta)}$$

(for $\beta > 5/3$), where the c_i's are finite constants depending upon β only. Thus, $\mathbb{E}\{C|X=x\} < \infty$ for $\beta > 5/3$. □

While theorem C1 implies that the algorithm halts with probability one, it is unfortunate that the tail of h is too large, and we still have $\mathbb{E}\{C\} = \infty$ for all values of β and the given choices for M and L. The squeeze steps suggested in a further section will finally eliminate this last obstacle.

Refinement 2: Without trigonometric integral functions

In the absence of the trigonometric integral functions $\mathcal{C}$ and $\mathcal{S}$, we need to adjust the algorithm above only slightly, as we will now show. We recall the following fast converging series:

$$\mathcal{S}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)\,(2n+1)!}$$

and

$$\mathcal{C}(t) = \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{(2n)\,(2n)!} \ .$$

This permits us to define approximations

$$\mathcal{S}_N(t) = \sum_{n=0}^{N} \frac{(-1)^n t^{2n+1}}{(2n+1)\,(2n+1)!}$$

and

$$\mathcal{C}_N(t) = \max\left(0, \sum_{n=1}^{N} \frac{(-1)^n t^{2n}}{(2n)\,(2n)!}\right) \ .$$

Thus, we generalize the definition of f_n as follows:

$$f_n^*(x) = \frac{1}{\pi L} \sum_{j=0}^{ML-1} e^{-\beta \mathcal{C}_N(j/L)} \cos(\beta \mathcal{S}_N(j/L) - (j/L)x) \ . \qquad (6)$$

LEMMA B3. *For M, L, N positive integers such that $N+1 \geq eM$, the above approximation satisfies, for all $x > 0$,*

$$|f_n^*(x) - f(x)| \leq R_n^*(x) \stackrel{\text{def}}{=} \frac{(2M\beta + Mx)}{\pi L} + \frac{8}{\pi x M^\beta} + \frac{\beta}{\pi e\, 2^{2N+2}} \ . \qquad (7)$$

PROOF. Clearly,

$$|f_n^*(x) - f(x)| \le |f_n^*(x) - f_n(x)| + |f_n(x) - f(x)| \stackrel{\text{def}}{=} I + II \ .$$

Since II is bounded by (3) in lemma B2, it suffices to bound I.

$$\begin{aligned}
I &= |f_n^*(x) - f_n(x)| \\
&\le \frac{M}{\pi} \sup_{0\le t\le M} \left| e^{-\beta\mathcal{C}(t)} \cos(\beta\mathcal{S}(t) - tx) - e^{-\beta\mathcal{C}_N(t)} \cos(\beta\mathcal{S}_N(t) - tx) \right| \\
&\le \frac{M}{\pi} \sup_{0\le t\le M} \left| e^{-\beta\mathcal{C}(t)} - e^{-\beta\mathcal{C}_N(t)} \right| + \frac{M}{\pi} \sup_{0\le t\le M} |\cos(\beta\mathcal{S}(t) - tx) - \cos(\beta\mathcal{S}_N(t) - tx)| \\
&\le \frac{\beta M}{\pi} \left(\sup_{0\le t\le M} |\mathcal{C}(t) - \mathcal{C}_N(t)| + \sup_{0\le t\le M} |\mathcal{S}(t) - \mathcal{S}_N(t)| \right) \\
&\le \frac{\beta M}{\pi} \sup_{0\le t\le M} \sum_{n=2N+2}^{\infty} \frac{t^n}{n\, n!} \\
&\le \frac{\beta M}{\pi} \sum_{n=2N+2}^{\infty} \frac{1}{n} \left(\frac{eM}{n} \right)^n \\
&\le \frac{\beta M}{\pi} \sum_{n=2N+2}^{\infty} \frac{1}{n} \left(\frac{1}{2} \right)^n \\
&\le \frac{\beta M}{\pi (N+1) 2^{2N+2}} \\
&\le \frac{\beta}{\pi e\, 2^{2N+2}}
\end{aligned}$$

if $N + 1 \ge eM$. This completes the proof. □

We apply the algorithm with $R_n(x)$ replaced by $R_n^*(x)$ and $f_n(x)$ replaced by $f_n^*(x)$. For its validity, we only require that $R_n^*(x) \to 0$ for all x. We can make $R_n^*(x)$ less than $2h(x)/2^n$ by taking L and M as in (4) and (5) (and thus, as in the full algorithm shown earlier), and picking N in the following manner:

$$N = \left\lceil \max\left(eM, \frac{1}{2}\left(n + \log_2\left(\frac{\beta}{4\pi e h(x)} \right) \right) \right) \right\rceil \ . \tag{8}$$

The computational burden per iteration grows as MNL instead of ML. Yet, with the choices of the three parameters suggested here, the algorithm yields exactly distributed random variates.

Purists may even object to the fact that we have the exp and cos functions in f_n^*. Even these can be avoided! In fact, it suffices to replace both by appropriately truncated Taylor series with K terms, with K yet another parameter that increases with n. Simple additional bounding not unlike that of lemma B3 allows one to get yet another value of the approximation error (to replace $R_n^*(x)$). As exp and cos are pretty standard, we will not proceed along these lines though.

Refinement 3: Exponential tail bounds and squeezing

Finally, a serious improvement is possible thanks to a squeeze step. The decision $[T < f(X)]$ in the algorithm is costly for large X because we need to take L proportional to $X/h(X)$, and h has indeed a fat tail. A squeeze step can be introduced in the algorithm, which can very quickly reject, that is, we find a bound $f(x) \leq h^*(x)$, and perform a quick rejection if $T > h^*(X)$. [Note: in the algorithm above, insert the lines "Accept $\leftarrow$ False" and "if $T < h^*(X)$ then" before the line "$n = 1$", and append the line "end if" after the statement "Accept $= [T \leq f_n(X) - R_n(X)]$".] If h^* has small tails, a nice speed-up will result. Note that it is not necessary at all to generate random variates from a density proportional to h^*, so we need only be concerned with its general shape.

LEMMA V3. *For the Vervaat family,*

$$f(t) \leq \frac{2\beta \mathbb{P}\{Z > t/2^{1/\beta}\}}{t} \ , \ t > 0 \ .$$

Lemma M2 indicates that the tail of Z drops off like that of a Poisson random variable. We will now derive an explicit Poisson tail inequality and other inequalities. In part D, we obtain the Poisson tail result of Goldie and Grübel (1995) mentioned in part C of lemma M2.

LEMMA V4. *Let Z be a Vervaat random variable and $\beta' = \max(1,\beta)$.*

A. *For all $t > 0$, and all $s > 0$,*

$$\mathbb{P}\{Z + 1 > t\} \leq e^{\rho(s) - st} \ ,$$

where $\rho(s) = 2\beta' s e^s/(1 - e^{-s})$.

B. *For all $t > 0$,*

$$\mathbb{P}\{Z + 1 > t\} \leq e^{8.61\beta' - t} \ .$$

C. *For $t \geq 12\beta'$,*

$$\mathbb{P}\{Z + 1 > t\} \leq e^{-(t/4)\log(t/4\beta')} \ .$$

D. *As $t \to \infty$,*

$$\mathbb{P}\{Z > t\} \leq \exp\left(-(1 + o(1))t \log t\right) \ .$$

Lemmas V3 and V4 may be combined to yield fast decreasing tail bounds on f. For example, part A yields the bound

$$f(t) \leq \frac{2\beta}{t} e^{\rho(s) - s - t/2^{1/\beta}} \ ,$$

valid for any choice $s > 0$. Part B yields

$$f(t) \leq h^*(t) \stackrel{\text{def}}{=} \frac{2\beta}{t} e^{8.61\beta' - 1 - t/2^{1/\beta}} \ . \tag{9}$$

We are finally ready for an overall analysis for the algorithm with squeezing based on (9), and with Simpson's rule for f_n, and the choices of $M = M_n$ and $L = L_n$ suggested there. The final complexity result is the following.

THEOREM C2. *Consider the algorithm with Simpson's rule in f_n, and with*

$$R_n(x) = \frac{\sigma M(x+m)^4}{\pi L^4} + \frac{8}{\pi x M^\beta} .$$

Introduce the squeeze step based on (9). Furthermore, choose M as in (4) and pick

$$L = \left\lceil \left(\frac{\sigma M(x+m)^4 2^{n+1}}{\pi h(x)} \right)^{\frac{1}{4}} \right\rceil .$$

Let C be the time taken by the algorithm to generate X. Then $C < \infty$ with probability one. Also, when $\beta > 5/3$, we have $\mathbb{E}\{C\} < \infty$.

PROOF. We note that X has density $h(x)/\int h$, and that the expected number of outer iterations of the algorithm is $\int h$. By Wald's lemma, the overall expected work is thus $\int h$ times the expected work in one iteration (regardless of acceptance or rejection). The effect of the squeeze step is to generate X from the density proportional to $\min(h, h^*)$. Let C be the complexity due to processing X. For a fixed x, we may still apply the bound on $\mathbb{E}\{C|X = x\}$ derived in the proof of theorem C1. Taking expected value with respect to the density of X (which is now proportional to $\min(h, h^*)$), we see that $\mathbb{E}\{C\}$ is finite if $\beta > 5/3$ and

$$\int \min(h(x), h^*(x)) \left(1 + \frac{1}{(xh(x))^{1/\beta}} + \frac{x+m}{h(x)^{1/4}(xh(x))^{1/(4\beta)}} + \frac{x+m}{h(x)^{1/4}(xh(x))^{5/(4\beta)}} \right) dx < \infty .$$

By considering integration over $[0,1]$ and $[1,\infty)$ separately, it is easy to see that the interval $[1,\infty)$ yields a finite contribution due to the exponential decrease of h^*. The integral on $[0,1]$ is finite when $\beta > 4/3$. □

Extensions

The methods developed here can be used for infinitely divisible distributions given in Lévy or Kolmogorov form, provided we also have enough information to be able to find a dominating density for rejection. For the Vervaat family, we were lucky to find a suitable Lévy form. For more general perpetuities, the problem is much harder. We are working on a solution based on the moment method of Devroye (1989) (which requires finite time computability of all moments, as in our case) in combination with Lagrange approximations of functions.

Appendix

PROOF OF LEMMA V3. Let g be the density of Z^β. Note that $f(z) = \beta z^{\beta-1} g(z^\beta)$. Since Z^β has a monotone density on the positive halfline, we have

$$\mathbb{P}\{Z^\beta > t^\beta/2\} \geq \int_{t^\beta/2}^{t^\beta} g(u)\,du \geq \frac{t^\beta g(t^\beta)}{2} = \frac{t^\beta f(t)}{2\beta t^{\beta-1}} = \frac{tf(t)}{2\beta} \; . \; \square$$

PROOF OF LEMMA V4. We replace $Z+1$ be a stochastically bigger random variable. Fix a small positive $u > 0$, and in the definition of $Z+1$, replace each $U_i^{1/\beta}$ by 1 if $U_i^{1/\beta} \geq 1-u$ and by $1-u$ otherwise. Call the new bigger random variable B. Let $T_1 < T_2 < T_3 < \cdots$ be the indices i for which $U_i^{1/\beta} \geq 1-u$. Then, a moment's thought shows that

$$B = T_1 + (1-u)(T_2 - T_1) + (1-u)^2(T_3 - T_2) + \cdots \; .$$

This is interesting, as the sequence $T_{i+1} - T_i, i \geq 0$, with $T_0 = 0$, is i.i.d. geometrically distributed. In fact, the parameter (success probability) of these geometric random variables is $p = \mathbb{P}\{U^{1/\beta} \leq 1-u\} = (1-u)^\beta$. If $s > 0$, and $G = T_1$ is geometric (p), then

$$\begin{aligned}
\mathbb{E}\left\{e^{sG}\right\} &= \sum_{j=1}^{\infty} e^{sj}(1-p)^{j-1}p \\
&= pe^s \sum_{j=0}^{\infty} (e^s(1-p))^j \\
&= \frac{pe^s}{1-e^s(1-p)} \quad (\text{if } e^s(1-p) < 1) \\
&= \frac{(1-u)^\beta e^s}{1-e^s(1-(1-u)^\beta)} \quad (\text{if } e^s(1-(1-u)^\beta) < 1) \; .
\end{aligned}$$

Note that $1-(1-u)^\beta \leq \max(1,\beta)u$. Define $\beta' = \max(1,\beta)$. So, for $\beta' u e^s < 1$, we have

$$\mathbb{E}\left\{e^{sG}\right\} \leq \frac{e^{-\beta u + s}}{1-\beta' u e^s} \; .$$

Take $u = (1-c^{-s})/(\beta' e^s)$ for $c > 1$, and obtain

$$\mathbb{E}\left\{e^{sG}\right\} \leq \frac{e^{-\beta u + s}}{1-\beta' u e^s} \leq (ce)^s \; .$$

By Chernoff's bounding method, for $s > 0$, and u as above,

$$\mathbb{P}\{B > t\} \leq \mathbb{E}\{e^{sB}\}e^{-st} = \prod_{i=0}^{\infty} \mathbb{E}\{e^{sG(1-u)^i}\}e^{-st} \leq \prod_{i=0}^{\infty} (ce)^{s(1-u)^i} e^{-st}$$

so that

$$\begin{aligned}
\log \mathbb{P}\{B > t\} &\leq -st + \sum_{i=0}^{\infty} \left(s(1-u)^i \log(ce)\right) \\
&= -st + \frac{s\log(ce)}{u} = -st + \frac{\beta' s e^s \log(ce)}{1-c^{-s}} \\
&= -st + \frac{2\beta' s e^s}{1-e^{-s}}
\end{aligned}$$

when we take $c = e$. The last bound is in the form $\rho(s) - st$ of part A of the lemma. Part B is obtained by picking $s = 1$ and noting that $2e^2/(e-1) < 8.61$. Part C is obtained by picking $s = \log(t/4\beta')$ so that when $t \geq 12\beta'$,

$$\log \mathbb{P}\{B > t\} \leq -st + \frac{(1/2)st}{1 - \frac{4\beta'}{t}} \leq -st + (3/4)st = -\frac{st}{4} \; .$$

Part D follows if we set $s = \log(t/(2\beta'(1 + \log(t+1))))$. □

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