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Random variate generators for the Poisson–Poisson and related distributions

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Abstract: Random variate generators are developed for the Poisson–Poisson (or generalized Poisson) distribution. The expected time per generated random variate is uniformly bounded over the parameter space. Random variate generation for related distributions is also discussed; these include the Abel and Ressel families, and a family introduced by Haight.

Keywords: Random variate generation, Poisson–Poisson distribution, Natural exponential family, Abel's distribution, Ressel's density, Haight's distribution, Lagrange distributions, Unimodality, Probability inequalities, The rejection method, Expected time analysis.

1. Introduction

A discrete distribution that has received quite a bit of attention in the past fifteen years is the Poisson–Poisson (or Lagrange double Poisson; or generalized Borel–Tanner; or generalized Poisson) distribution defined by

$$p_n = \frac{p(\lambda n + p)^{n-1} e^{-(\lambda n + p)}}{n!}, \qquad n \ge 0, \tag{1}$$

where $\lambda \in [0, 1]$ and $p > 0$ are shape parameters. It was first introduced by Consul and Jain (1973), and is only a special case of the large family of Lagrange distributions, of which a list of examples can be found in Consul and Shenton (1972), and whose properties are studied in Consul and Shenton (1973, 1975) and Jain (1974). See also the surveys by Johnson and Kotz (1982) or Patil et al. (1984, p. 23).

As (extreme) special cases we have:

A. Case $\lambda = 0$: the **Poisson** distribution with parameter p.
B. Case $\lambda = 1$: the **Abel** distribution with parameter p. Many distributions can be derived by Abel's generalizations of certain expansions; for the binomial

expansion, see e.g. Consul (1974) or Consul and Mittal (1975). One such Abel distribution, which we will call "the" Abel distribution (to keep the terminology simple), is defined by the the Poisson–Poisson distribution with parameter $\lambda = 1$. Natural exponential families play an important role in mathematical statistics (see Barndorff–Nielsen (1978) or Morris (1982)). Recently, Letac and Seshadri (1987) identified four natural exponential subfamilies as the only ones on $(0, \infty)$ satisfying the property that $E(1/X) = (a/E(X)) + b$ for some constants a, b. These are the gamma, inverse gaussian, Ressel and Abel families. These distributions are "primitive" in the sense that they are not originally defined in terms of standard operations such as convolution, compounding, contamination, or transformations. This could sometimes create problems when one wishes to generate random variables with such distributions, since we can't usually combine other random variables in a straightforward manner. It is well-known that for the gamma family several very good methods are available, see e.g. Ahrens and Dieter (1974, 1982), Greenwood (1974), Cheng (1977), Marsaglia (1977), Vaduva (1977), Kinderman and Monahan (1978, 1979), Cheng and Feast (1979, 1980), Schmeiser and Lal (1980), Ahrens, Kohrt and Dieter (1983), Best (1983), Barbu (1987), or the surveys in Tadikamalla and Johnson (1981) and Devroye (1986). The same is true for the inverse gaussian family (Michael, Schucany and Haas (1976), Padgett (1978)). Unfortunately, the same cannot be said for the less known Ressel family studied by Letac and Mora (1987), and the Abel family studied by Mora (1978).

C. Case $\lambda = p$: we obtain a discrete distribution that defines the mixture of gamma densities into which Haight's density can be decomposed. We will call this the **Haight mixture** distribution. Recently, Letac and Seshadri (1988) drew the attention to a family first proposed in Haight's survey (Haight, 1961). In a form suitable for consumption by us, and without a superfluous scale parameter, Haight's distribution can best be defined in terms of its density

$$f(x) \triangleq \frac{1}{a}\, \mathrm{e}^{-xe^a/a} \sum_{n=0}^{\infty} \frac{x^n (n+1)^{n-1}}{n!^2}, \qquad x > 0,$$

where $a \in (0, 1]$ is a parameter. The case $a = 1$ was not considered by Haight in his entry 8.89 on p. 54, but it is easy to verify that even for $a = 1$, f is a density. The density can be considered as a mixture

$$f(x) = \sum_{n=0}^{\infty} p_n g_{n,ae^{-a}}(x),$$

where $g_{n,ae^{-a}}$ is the gamma density

$$g_{n,ae^{-a}}(x) = \frac{x^n\, \mathrm{e}^{-x/(ae^{-a})}}{(ae^{-a})^{n+1} n!}, \qquad x > 0,$$

and p_n, $n \geq 0$, is a probability vector

$$p_n = \frac{(ae^{-a})^{n+1}(n+1)^{n-1}}{an!},$$

called the **Haight mixture distribution.** Note that p_n is the Poisson–Poisson distribution with parameters $\lambda = p = a$. Generation of a variate with density f proceeds best via the mixture method: first, we generate a random variate X having probability vector p_n, then generate a gamma random variate Y with parameter $X+1$, and finally return $ae^{-a}Y$.

Some properties of the Poisson–Poisson distribution, along with its genesis, can be found in section 2. It is immediately apparent that there is no straightforward manner in which Poisson–Poisson random variates can be obtained by simple combinations of more primitive random variables such as Poisson, binomial, uniform or normal random variables. The different possible approaches for generating such random variates are discussed in section 3. From a computational perspective, we would prefer to have a method with expected time per variate uniformly bounded over the parameter space. Also, the algorithms should not be too lengthy. There are general methods for unimodal distributions with known mode, mean and variance, developed e.g. in Devroye (1986, pp. 493–495). These techniques have to be slightly adapted however when the location of the mode is unknown, and in any case do not yield uniformly bounded computation times (see section 4). We are thus led to consider algorithms that are specifically designed for the present family (section 5). These algorithms have uniformly bounded computation times over the entire parameter space. If $E(N)$ is the expected number of iterations in the respective rejection algorithms, we will among other things obtain the following results for the important subfamilies identified above:

A. The Poisson–Poisson generator of section 5.2 is uniformly fast for the Poisson (p) distribution when $p \geq 3$; moreover, it is asymptotically optimal in the sense that $E(N) \to 1$ as $p \to \infty$. This implies that for large p, one Poisson random variate is obtained at the cost of about two uniform random variates and one normal random variate. However, the algorithm is not short, the number of constants that have no computed in a set-up is considerable, and the resulting code is not short. For shorter code, we refer to the method of section 4, for which $E(N) \to 2.34\ldots$ as $p \to \infty$. For parameter $p < 1$, the rejection algorithm with polynomially decreasing bound of section 5.1 could be employed.

B. For the Abel distribution, the algorithm of section 5.3 has uniformly bounded expected time for all $p > 0$. As $p \to \infty$, we have $E(N) \to 2.4811\ldots$

C. Since the Haight mixture distribution is nonincreasing (Lemma P1), random variate generation can proceed by rejection from the polynomial bound described in section 5. In Theorem C1, we will show that

$$E(N) = \mathrm{e}^{-p} + pe^{2-p}\sqrt{\frac{2}{\pi}}.$$

This is maximal at $p = 1 - \mathrm{e}^{-2}\sqrt{\pi/2}$, and the maximal value is $2.569795364\ldots$

Finally, in section 6, a family of densities coined Ressel densities by Letac and Seshadri (1987) is considered. These densities are in form similar to the discrete Abel distributions. We develop a generator which with probability tending to one as the parameter tends to ∞ requires only one normal and two uniform random variates per generated Ressel variate. Our notation includes $\lceil . \rceil$, $\lfloor . \rfloor$ (rounding to the nearest larger and smaller integers), $\triangleq$ (definition), $:=$ (definition), $(.)_+$ (maximum of . and 0), $\uparrow$, $\downarrow$ (monotone convergence upwards and downwards).

2. Properties of the distribution

A discrete distribution is log-concave on an interval of indices if $p_{n-1}p_{n+1} \le p_n^2$ for all the indices n in this interval. Equivalently, p_{n+1}/p_n is $\downarrow$ for n in the interval.

Lemma P1. Log-concavity. If $\lambda = 0$, the distribution is log-concave. If $\lambda > 0$, the distribution is log-concave on $0, \ldots, n$, where $n + 1 \le p(p-\lambda)/(2\lambda^2)$.

Lemma P2. Moments. (Consul and Shenton, 1972). When $\lambda < 1$, all moments exist, and, in particular, the mean is $p/(1-\lambda)$, and the variance is $p/(1-\lambda)^3$.

Lemma P3. Unimodality and monotonicity. For $p \le 1 + \lambda$, the distribution is monotone $\downarrow$. For $p \ge \max(1+\lambda, \lambda/(1-\lambda))$, $\lambda \in [0, 1)$, the distribution is unimodal. In the latter case, at least one peak located between $\lceil (p-1)/(1-\lambda) \rceil$ and $\lceil (p-d)/(1-\lambda) \rceil$, where $d = (1+\lambda^2)/(1-\lambda)$. Also, for $p \ge \max(1+\lambda, 2\lambda/(1-\lambda))$, the distribution is log-concave on $[0, \lceil (p-1)/(1-\lambda) \rceil]$.

Lemma P4. Genesis and representation. (Consul and Shenton, 1972, 1974). A Poisson–Poisson random variable X is distributed as $\sum_{i=1}^{N} Z_i$, where $N, Z_1, \ldots, Z_n, \ldots$ are independent random variables; N is a Poisson random variable with parameter p; and the Z_i's are iid random variables with moment generating functions $s(u) = E(u^{Z_1})$ given by the solution of the equation $u = s/\exp(\lambda(s-1))$. Furthermore, $EZ_1 = 1/(1-\lambda)$.

From the previous representation, it follows that the sum of a Poisson–Poisson (λ, p) and an independent Poisson–Poisson (λ, q) random Poisson–Poisson $(\lambda, p+q)$.

From Stirling's approximation for $\Gamma(n+1)$, we have

Lemma P5. General factorial-less bound.

$$p_n \le \frac{p(\lambda n + p)^{n-1} \mathrm{e}^{n+1-\lambda n - p}}{\sqrt{2\pi}\,(n+1)^{n+1/2}} \triangleq \mathrm{e}^{\rho_n}, \qquad n \ge 0,$$

where $\rho_n = 1 - p + \log(p) - (1/2)\log(2\pi) + (n-1)(\log(\lambda n + p) - \log(n+1)) - (3/2)\log(n+1) + (1-\lambda)n$.

Lemma P6. Polynomial bound. *For all* λ, p, $p_0 = e^{-p}$. *For all* $n > 0$, *we have*

$$p_n \le \frac{pe^{2(1-\lambda)+[2(\lambda-p)/(n+1)]}}{\sqrt{2\pi}\,(n+1)^{3/2}} \le pe^{2(1-\lambda)+[2(\lambda-p)/(n+1)]}\sqrt{\frac{2}{\pi}}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$

$$\le pe^{2-\lambda-\min(\lambda,p)}\sqrt{\frac{2}{\pi}}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right).$$

3. Design of a generator

The representation of Lemma P4 is of little practical value. Even if we could generate the individual Z_i's at unit cost, the total expected cost would be p, and is therefore not uniformly bounded over the parameter space. However, the representation is useful when we try to visualize the parameter space. For example, when p is large with respect to $1/(1-\lambda)$, the influence of the random Poisson number of terms dominates the influence of the distribution of Z_1. When p is small with respect to $1/(1-\lambda)$, the distribution of Z_1 becomes dominant.

For families of distributions, truly efficient generators can normally be designed by the rejection method with as dominating density a suitably modified limit density. This is bound to cause problems here, since, as is shown by Consul and Shenton (1973), for λ fixed and $p \to \infty$, the limit law is normal, and for $\lambda \to 1$ in such a way that $p(1-\lambda) \to c$ as $p \to \infty$, the limit law is inverse gaussian with parameter c. At the very least, this means that a normal dominating curve is not efficient in all cases.

For simple one-parameter families, the design of a generator is usually not too testing. For more complex families, it is sometimes helpful to borrow from the general algorithms that are available in the literature. For discrete log-concave distributions for example, the algorithms of Devroye (1987) have uniformly bounded times; they do require however that the mode be known. If the mode is not known, a small adaptation can be implemented that works roughly as follows: find four integers, two close to but at opposite sides of the mode, and two close to the mode plus and minus about a standard deviation. At these points, geometric dominating densities can be found that agree with the original density at two neighboring points (by log-concavity). Rejection can now proceed based upon a mixture of four pieces of geometric dominating densities. Such a strategy would work here with one proviso: for the extreme right tail of the distribution, a fifth dominating curve is required, because the Poisson–Poisson family is not log-concave in its right tail. At this point, we are staring at a rather complicated endeavour.

Another general strategy was proposed in Devroye (1986, pp. 493–495) for unimodal densities with known mode, mean and variance. The algorithm is shown there to yield uniform speeds for the binomial and Poisson families, and should thus be useful here. Interestingly, it can be modified to make up for the fact that we do not know the location of the mode. Unfortunately, the algorithm is not uniformly fast over our parameter space.

The approach followed below for the main portion of the parameter space consists simply of obtaining a good normal dominating curve with special treatment for the tails. Slightly different strategies are followed for two extreme regions of the parameter space.

4. A universal rejection method

The situation is the following: a unimodal discrete probability distribution with probabilities p_n is given with known mean μ, variance σ^2 and mode m. Furthermore, it is known that $\sup_n p_n \le M$. A general algorithm for this situation is presented in Devroye (1986, pp. 493–495) (in the algorithm on p. 495, the constant $\rho/(3\rho + M)$ should be replaced by $3\rho/(3\rho + M)$ however). The expected number of iterations is

$$M + 3\left(3\left(\sigma^2 + (\mu - m)^2\right)\right)^{1/3} M^{2/3}.$$

In order to be able to apply the said algorithm, it is necessary to compute m and M (or a good upper bound for M) in a preprocessing step, since closed analytical expressions for these quantities are not known, except in special cases. Also, for $\lambda = 1$, both μ and σ are infinite, which confirms that the expected number of iterations is not uniformly bounded over the entire parameter space; actually, the performance deteriorates with decreasing values for $p(1-\lambda)$. However, the method is applicable for some important special cases. Consider for example the Poisson distribution with parameter p. We have $\mu = \sigma^2 = p$, $m = \lfloor p \rfloor$ and $M = p_m \le 1/\sqrt{2\pi m}$ (for the last inequality, see Devroye (1986, p. 506)). Here, no preprocessing is necessary, and the expected number of iterations is bounded by

$$M + 3(3(p+1))^{1/3} M^{2/3}.$$

As $p \to \infty$, the upper bound is $o(1) + (81/2\pi)^{1/3}$, and the constant is about 2.344777925... The expected time is uniformly bounded for $p \ge 1$, and the computer code is shorter than for most other uniformly fast Poisson generators. For longer and sometimes more efficient methods, consult Ahrens and Dieter (1980, 1982), Ahrens, Kohrt and Dieter (1983), Atkinson (1979), Devroye (1981, 1986), Kachitvichyanukul (1982), and Schmeiser and Kachitvichyanukul (1981).

5. The Poisson–Poisson family

We consider three regions in the parameter space:

A. The region of monotonicity: $p \le 1 + \lambda$. The algorithm developed below is applicable however in all situations, and has uniformly bounded times whenever $p \le c$ for some constant c.

B. The Poisson side of the parameter space: $p \geq \max(3, 2\lambda/(1-\lambda))$. The distribution is unimodal, and includes the Poisson distribution for $\lambda = 0$, $p \geq 1$.
C. The Abel side of the parameter space: $p \geq 1 + \lambda$, $p \leq 2\lambda/(1-\lambda)$.

5.1. The region of monotonicity

We can use rejection based upon the inequalities given in Lemma P6: $p_0 = e^{-p}$ and for $n > 0$,

$$p_n \leq p e^{2-\lambda-\min(\lambda,p)} \sqrt{\frac{2}{\pi}} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right).$$

The upper bound is in a form convenient for the inversion method. Indeed, the random variate $X \leftarrow \lfloor 1/U^2 \rfloor$ where U is uniform on [0, 1] satisfies $P(X \geq n) = P(1 \geq nU^2) = \sqrt{1/n}$ for $n \geq 1$. Hence, $P(X = n) = \sqrt{1/n} - \sqrt{1/(n+1)}$ as required. The details are all provided in the algorithm below. The algorithm is theoretically applicable in the entire parameter space, but only recommended when $p \leq 1 + \lambda$.

Rejection algorithm with polynomial bound

[SET-UP]
$p_0 \leftarrow e^{-p}$, $b \leftarrow p e^{2-\lambda-\min(\lambda,p)} \sqrt{2/\pi}$.

[GENERATOR]
Generate a uniform [0, 1] random variate U.
REPEAT

IF $U \leq \dfrac{p_0}{p_0 + b}$

THEN $X \leftarrow 0$ and Accept $\leftarrow$ True.
ELSE
Generate iid uniform [0, 1] random variates V, W.

Set $X \leftarrow \lfloor 1/W^2 \rfloor$.

Set Accept $\leftarrow \left[Vb\left(\dfrac{1}{\sqrt{X}} - \dfrac{1}{\sqrt{X+1}} \right) \leq p_X \right]$.

UNTIL Accept.
RETURN X.

The notation p_X is used for p_n at $n = X$ (see (1)). The evaluation of p_X can be done efficiently (i.e., without having to evaluate $\log(n!)$ or without having to worry about cancelation errors in $(\lambda n + 1)^{n-1}/n!$) by making use of squeeze steps obtained by bounds for the gamma function similar to those employed in

Lemma P5. For more on the avoidance of factorials, see chapter X of Devroye (1986).

Theorem C1. Complexity. Assume that the above algorithm based upon polynomial rejection is used. Then

$$E(N) = e^{-p} + pe^{2-\lambda-\min(\lambda,p)}\sqrt{\frac{2}{\pi}}\ .$$

We also have

$$\sup_{p \le 1+\lambda} E(N) = 1/e + e^2\sqrt{2/\pi}\ ,$$

where the supremum is reached for $p=1$, $\lambda=0$. Finally, $E(N) \le 1 + Ce^2\sqrt{2/\pi}$ whenever $p \le C$.

5.2. The Poisson side of the parameter space

The probabilities p_n will be bounded by some function $g(x)$ uniformly on $n \le x \le n+1$. This will allow us to develop rejection algorithms from continuous univariate distributions. The prime candidate for bounding is the normal density, with exponential tails added on both ends. If Y is a random variate with density proportional to g and U is a uniform [0, 1] random variable, then it suffices to generate pairs (Y, U) until for the first time $Ug(Y) \le p_X$ where $X = \lfloor Y \rfloor$. The random variable X has the sought discrete distribution p_n. The main body of the dominating curve is the normal part. The bound, given below in Lemma C2, is such that the area under the global dominating curve tends to 1 as $p \to \infty$, while $\lambda > 1$ is held fixed.

Lemma C2. Let $n \ge 0$ be integer, and assume that $p > \lambda$, and $(p-\lambda)/(1-\lambda+\delta) \le n+1 \le (p-\lambda)/(1-\lambda-\epsilon)$, where $0 < \epsilon < 1-\lambda$, $p-1 > \delta > 0$. Then, for all $n \le x \le n+1$, we have

$$p_n \le g(x) \triangleq G(\sqrt{2\pi}\,\sigma)^{-1}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where

$$\phi \triangleq \frac{p\delta(2+\delta-2\lambda) + (1+\delta)(1-\lambda)^2 - \lambda(1-\lambda+\delta)^2}{2(p-1-\delta)},$$

$$\mu \triangleq \frac{p-\lambda}{1-\lambda},\quad \sigma^2 \triangleq \frac{(1+\delta)(p-\lambda)}{(1-\lambda-\epsilon)(1-\lambda)^2},$$

and

$$G \triangleq \int_0^\infty g = \frac{p(1-\lambda-\epsilon)\sqrt{1+\delta}}{(p-\lambda)(1-\lambda)(1-\epsilon)^2}\,e^{(\phi/(1+\delta))}.$$

Lemma C3. Assume that n, x, p, ϵ and δ are as in Lemma C2. For $n+1 \geq (p-\lambda)/(1-\lambda-\epsilon)$, and $n \leq x \leq n+1$, we have

$$p_n \leq h_r(x) \triangleq \frac{p(1-\lambda-\epsilon)^{3/2}}{\sqrt{2\pi}(p-\lambda)^{3/2}} e^{-(1-2(1-\lambda-\epsilon)/(p-\lambda))(\epsilon/2)(1-\lambda)(x-\mu)+2(1-\lambda)}.$$

Define $t_r = \lceil (p-\lambda)/(1-\lambda-\epsilon) - 1 \rceil$. Then

$$H_r \triangleq \int_{t_r}^{\infty} h_r = \left[2p(1-\lambda-\epsilon)^{3/2} e^{2(1-\lambda)}\right] \times \left[\sqrt{2\pi}(p-\lambda)^{3/2}\left(1-\frac{2(1-\lambda-\epsilon)}{p-\lambda}\right)\epsilon(1-\lambda)\right]^{-1} \times e^{-(1-2(1-\lambda-\epsilon)/(p-\lambda))(\epsilon/2)(1-\lambda)(t_r-\mu)}.$$

For $n+1 \leq (p-\lambda)/(1-\lambda+\delta)$, and $n \leq x \leq n+1$, we have

$$p_n \leq h_l(x) \triangleq \frac{p}{\sqrt{2\pi}} e^{[\delta(1-\lambda)]/[2(1+\delta)](x+1-\mu)}.$$

Define $t_l = \lceil (p-\lambda)/(1-\lambda+\delta) - 1 \rceil$. Then

$$H_l \triangleq \int_{-\infty}^{t_l} h_l = \frac{2p(1+\delta)}{\sqrt{2\pi}\,\delta(1-\lambda)} e^{[\delta(1-\lambda)]/[2(1+\delta)](t_l+1-\mu)}.$$

From Lemmas C2 and C3, we can construct a global bound as follows: assume that $p > \lambda$, $1-\lambda > \epsilon > 0$ and that $0 < \delta < p-1$; then

$$p_n \leq \begin{cases} h_l(x) & n < t_l \\ g(x) & t_l \leq n < t_r, \quad n \leq x < n+1, \\ h_r(x) & t_r \leq n \end{cases}$$

where h_l, h_r and g are defined in the two Lemmata. The rejection method can now be implemented as follows:

Poisson–Poisson generator

[INITIALIZATION]

The algorithm below should be used only when $0 < \epsilon < 1-\lambda$ and $p - 1 > \delta > 0$, where ϵ and δ are parameters chosen by the user. E.g., with the choices proposed in Theorem C2, it is additionally required that $p \geq \max(3, 2\lambda/(1-\lambda))$.

Compute ϕ, σ, μ and G as in Lemma C2. Compute t_l, t_r, H_l and H_r as in Lemma C3. Let g be as in Lemma C2, and let h_l and h_r be as in Lemma C3.

[GENERATOR]

REPEAT

Generate a uniform [0, 1] random variate U.

IF $U < \dfrac{G}{G - H_l + H_r}$ THEN

Generate a normal random variate N, and set $Y \leftarrow \mu + \sigma N$.
IF $Y \geq t_r$ or $Y < t_l$
  THEN Accept ← False
  ELSE
    Generate a uniform [0, 1] random variate V.
    $X \leftarrow \lfloor Y \rfloor$.

$$\text{Accept} \leftarrow \left[V_g(Y) \leq p_X \right].$$

ELSE IF $U < \dfrac{G + H_l}{G + H_l + H_r}$ THEN

Generate an exponential random variate E,

$$\text{and set } Y \leftarrow t_l - \frac{2E(1+\delta)}{\delta(1-\lambda)}.$$

IF $Y < 0$ THEN Accept ← False
  ELSE
    Generate a uniform [0, 1] random variate V.
    $X \leftarrow \lfloor Y \rfloor$.

$$\text{Accept} \leftarrow \left[Vh_l(Y) \leq p_X \right].$$

ELSE
  Generate an exponential random variate E, and set

$$Y \leftarrow t_r + \frac{2E}{\left(1 - \dfrac{2(1-\lambda-\epsilon)}{p-\lambda}\right)\epsilon(1-\lambda)}.$$

  Generate a uniform [0, 1] random variate V.
  $X \leftarrow \lfloor Y \rfloor$.

$$\text{Accept} \leftarrow \left[Vh_r(Y) \leq p_X \right].$$

UNTIL Accept
RETURN X.

The complexity of the algorithm can to some extent be measured by the expected number of iterations, $G + H_l + H_r$. This assumes that evaluations of g, h_l and h_r can be done at unit cost (see remarks elsewhere about the evaluation of factorials in constant expected time). The expected number of iterations should be minimized with respect to δ and ϵ. Unfortunately, this seems to be impossible to achieve in a closed analytical form. Basically, we want to find ϵ and δ such that two goals are achieved:

A. For each fixed $\lambda \in [0, 1)$, as $p \uparrow \infty$, the expected number of iterations tends to one, i.e. the algorithm has an asymptotically optimal dominating curve. As pointed out earlier, this implies that the dominating curve defines the limit law.
B. Uniformly over the Poisson side of the parameter space, the expected number of iterations remains bounded.

Theorem C2. Complexity. If in the algorithm described above, we take δ and ϵ such that for each fixed $\lambda \in [0, 1)$, as $p \uparrow \infty$, $\delta \to 0$, $\delta\sqrt{p/\log p} \to \infty$, $\epsilon \to 0$, and $\epsilon\sqrt{p} \to \infty$, then $G \to 1$, $H_r \to 0$ and $H_l \to 0$. In particular, property A holds.

For the algorithm given above, when employed with $p \geq 3$, $(p-\lambda)(1-\lambda) \geq \lambda$ (i.e. $p \geq 2\lambda/(1-\lambda)$), properties A and B hold if we take

$$\epsilon = \frac{1-\lambda}{(2+(p-\lambda)(1-\lambda))^{1/3}}, \quad \delta = \frac{(1-\lambda)^{2/5}}{(2+(p-\lambda)(1-\lambda))^{1/3}}.$$

5.3. The Abel side of the parameter space

This brings us finally to the Abel side of the parameter space, on which the universal rejection algorithm of section 4 or the algorithm of the previous section is either invalid, or not applicable, or not recommended. For example, the universal bounding function alluded to in section 4 has tails that decrease as x^{-3}; yet, we know that the Abel distribution (occurring at $\lambda = 1$) has a right tail that decreases as $n^{-3/2}$. Also, its mean is infinite. Thus, there is no hope of applying the universal rejection method in the entire parameter space. Observe that p_n is not log-concave, so that the universal algorithms for discrete log-concave distributions given in Devroye (1987) can not be applied. Fortunately, p_n is log-concave for all n smaller than some threshold value. This opens the door for an algorithm with an exponential dominating curve to the left and perhaps a polynomially decreasing dominating curve to the right.

The strategy followed by us came about as follows. First we designed a uniformly fast generator for the Abel distribution and optimized the parameter settings. This design called for a two-tier rejection algorithm, with the polynomial bound of Lemma P6 used on the right tail, and a geometrically increasing dominating function used on the left portion. The best threshold point was about equal to a constant times p^2. The geometrically increasing left portion gives a good fit since to the left of the threshold point the probability vector is log-concave. The region of space with $p \geq 1 + \lambda$, $p(1-\lambda) \leq 2\lambda$ looks a bit like a spiky infinite wedge. It is convenient to consider asymptotics within the peak by letting $p \to \infty$ such that $p(1-\lambda) \to c$, where $c \in [0, 2]$ is a constant. We then repeated the design with optimal parameter settings that depend upon c. The parameter setting that is optimal for the Abel distribution (case $\lambda = 0$, $c = 0$) turns out to be acceptable throughout this side of the parameter space. In

Theorem C3, we will show that the resulting algorithm has uniformly bounded expected time. We begin with the necessary upper bounds:

Lemma C4. *We have* $p_0 = \mathrm{e}^{-p}$. *Also, for all* $n > 0$,

$$p_n \le p e^{2-\lambda-\min(\lambda,p)} \sqrt{\frac{2}{\pi}} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right).$$

The sequence ρ_n *(defined in Lemma P5) is not concave on the real line for any pair* (p, λ) *with* $\lambda > 0$. *Assume that* $p \ge 1 + \lambda$. *Then, for real* u, ρ_u *is concave for* $u \le v \triangleq 2(p^2 - \lambda p - 3\lambda^2)/(3\lambda^2)$. *The constant* v *is nonnegative when* $p \ge (1/2)(\lambda + \sqrt{12\lambda + \lambda^2})$, *which in turn holds when* $p \ge 3$. *If* $v > 0$ *and* $n \le u \le v$, *where* n *is integer, then* $p_n \le \mathrm{e}^{\rho_u + (n-u)\rho_u'}$.

The generator we propose below uses rejection from two different dominating curves, one valid on $\{0, \dots, t\}$, and one on $\{t+1, \dots\}$, where the integer t is a threshold value. It is advantageous to choose $t = \lfloor \alpha \max(v, 0) \rfloor$, where $\alpha \in (0, 1]$ is a constant possibly depending upon the product $p(1-\lambda)$, and v is the value defined in Lemma C4. The parameter α will be picked so as to minimize the total area under the dominating curves. For the leftmost piece, we use

$$p_n \le \mathrm{e}^{\rho_t + (n-t)\rho_t'} = q_l (1-q) q^{t-n}, \qquad n \le t,$$

where $q_l = \mathrm{e}^{\rho_t}/(1-q)$ and $q = \mathrm{e}^{-\rho_t'}$. We note that $p \ge 1 + \lambda$ is needed for this inequality to hold. For smaller values of p, we simply omit the left hand part (see algorithm below). The area under the left dominating curve is

$$\sum_{i=0}^{t} \mathrm{e}^{\rho_t + (i-t)\rho_t'} = q_l \sum_{i=0}^{t} (1-q) q^i,$$

where $q = \mathrm{e}^{-\rho_t'}$. If we extend the geometric distribution to infinity, then the area under the geometric dominating curve is simply q_l. This is the area that should be counted if we use rejection from the un-truncated geometric distribution.

For the right tail, we employ rejection from a polynomially decreasing distribution, based on the polynomial inequality given in Lemma C4. The upper bound is in a form convenient for the inversion method. Indeed, the random variate $X \leftarrow \lfloor (t+1)/U^2 \rfloor$ where U is uniform on [0, 1] satisfies $P(X \ge n) = P(t+1 \ge nU^2) = \sqrt{(t+1)/n}$ for $n \ge t+1$. Hence, $P(X = n) = \sqrt{(t+1)/n} - \sqrt{(t+1)/(n+1)}$ as required. Also, the area under the upper bound (i.e., the sum of the individual elements for all $n \ge t+1$) telescopes easily to

$$q_r \triangleq p e^{\max(1-p,0)} \sqrt{\frac{2}{\pi(t+1)}} .$$

Before turning to the choice of α and the analysis of $E(N)$, it is helpful to give the algorithm in its entirety:

Poisson–Poisson generator

[SET-UP]
We assume that $p \geq 1+\lambda$ and that $p(1-\lambda) \leq 2\lambda$. let $\alpha \in (0,3/7)$ be a constant (the value 0.2746244084... is recommended in Theorem C3 below). Compute $v \leftarrow (2/3)(p^2 - \lambda p - 3\lambda^2)/\lambda^2$. Set $t \leftarrow \lfloor \alpha \max(v, 0) \rfloor$. If $p < 1+\lambda$ or $p(1-\lambda) > 2\lambda$, set $t \leftarrow 0$. Compute $b = pe^{2-\lambda-\min(\lambda, p)}\sqrt{2/\pi}$. Compute $q_r = b/\sqrt{t+1}$. If $t > 0$ then compute $q = e^{-p_t'}$ (see proof of Lemma C4 for expression), and $q_l = e^{p_t}/(1-q)$ (see Lemma P5 for definition). If $t = 0$ define $q_l = e^{-p}$.

[GENERATOR]
Generate a uniform [0, 1] random variate U.
REPEAT

IF $U \leq \dfrac{q_l}{q_l + q_r}$

THEN IF $t = 0$
  THEN X $\leftarrow$ 0 and Accept $\leftarrow$ True.
  ELSE
    Generate an exponential random variate E.
    Set $X \leftarrow t - \lfloor -E/\log(1-q) \rfloor$.
    IF $X > 0$
      THEN Set Accept $\leftarrow$ False.
      ELSE Generate a uniform [0, 1] random variate V.
      Set Accept $\leftarrow [V q_l q^{t-X}(1-q) \leq p_X]$.
ELSE
  Generate iid uniform [0, 1] random variates V, W.
  Set $X \leftarrow \lfloor (t+1)/W^2 \rfloor$.
  Set Accept $\leftarrow \left[Vb\left(\dfrac{1}{\sqrt{X}} - \dfrac{1}{\sqrt{X+1}} \right) \leq p_X \right]$.

UNTIL Accept.
RETURN X.

Theorem C3. Complexity. Let $\alpha \in (0,3/7)$ and $c \in [0, 2]$ be constants. If $p \to \infty$ in such a way that $p(1-\lambda) \leq 2\lambda$ and $p(1-\lambda) \to c$, then the algorithm given above satisfies

$$\lim_{p \to \infty} E(N) = \sqrt{\frac{3}{\pi\alpha}} + \sqrt{\frac{\alpha}{3\pi}} \frac{e^{-(1-2\alpha c/3)^2/(4\alpha/3)}}{1-\alpha-2\alpha c/3 - \frac{1}{2}(1-2\alpha c/3)^2}.$$

For $c = 0$, *this limit is minimal for* $\alpha = 0.2746244084\ldots$ *and takes the value* $2.4811500082\ldots$ *in that case. Finally, for any* $\alpha \in (0,3/7)$,

$$\sup_{(p,\lambda):\, p \geq 1+\lambda,\, p(1-\lambda) \leq 2\lambda} E(N) < \infty .$$

In Theorem C3, we have shown that the design is robust, since the expected time of the rejection algorithm is uniformly bounded over all p, λ on the Abel side of the parameter space. The parameter optimization was done for the Abel distribution (case $\lambda = 0$) only. The asymptotic performance of the algorithm can be improved by making α a function of c; for finite p, one then replaces $\alpha(c)$ by $\alpha(p(1-\lambda))$. We will not pursue this any further. The asymptotic value of $E(N)$ shows that there is room for improvement even for the Abel distribution, perhaps by adding in a third dominating curve, or by basing rejection on the inverse gaussian limit law. We have opted here for an algorithm that requires little work.

6. Ressel's class of densities

The **Ressel density** is defined by

$$f(x) = \frac{p x^{x+p-1}\, \mathrm{e}^{-x}}{\Gamma(x+p+1)}, \qquad x > 0,$$

where $p > 0$ is the parameter.

6.1. Inequalities for the Ressel density

We must first replace the gamma function in the definition of the density by a good estimate. This is done by a form of Stirling's formula, found e.g. in Whittaker and Watson (1927, p. 253), which states that for all $u > 0$,

$$\Gamma(u) = \left(\frac{u}{e}\right)^{u} \sqrt{\frac{2\pi}{u}}\; \mathrm{e}^{(\theta/12u)},$$

where $\theta \in [0, 1]$ is a number possibly depending upon u. Using this, we obtain without work

Lemma R1. *In all cases,*

$$f(x) \leq h(x) \triangleq \frac{p e^{p+1}}{\sqrt{2\pi}} \left(\frac{x}{p+x+1}\right)^{p+x} \frac{1}{x\sqrt{p+x+1}} .$$

Lemma R2. *In all cases,*

$$h(x) \leq \frac{p}{x\sqrt{2\pi}\,\sqrt{p+x+1}}\; \mathrm{e}^{-[(p+1)^2/2x]+[(p+1)/x]+[(p+1)^2 p/2x^2]} .$$

Theorem R1. *Let g be the density of the random variable* $Z = (p+1)^2/(2X)$, *where X has density f. Then*

$$g(z) \le A(z, p)\frac{\mathrm{e}^{-z}}{\sqrt{\pi z}},$$

where

$$A(z, p) \triangleq \frac{p}{1+p}\exp\left(\frac{2z}{p+1} + \frac{2pz^2}{(p+1)^2}\right).$$

In particular,

A. *For every fixed* $z > 0$, *as* $p \to \infty$, *we have* $g(z) \to \mathrm{e}^{-z}/\sqrt{\pi z}$. *Also,* $\int | g(z) - \mathrm{e}^{-z}/\sqrt{\pi z} \mid \to 0$ *as* $p \to \infty$. *In other words, Z is asymptotically gamma with parameter* $1/2$.
B. *For all* $z \le c(p+1)$, *we have*

$$g(z) \le \frac{pe^{2c}}{(1+p)\sqrt{1-\dfrac{2pc}{p+1}}}\left(1-\frac{2pc}{p+1}\right)\frac{e^{-z(1-2pc/(p+1))}}{\sqrt{\pi z\left(1-\dfrac{2pc}{p+1}\right)}}.$$

For values of p near zero, the following inequality will be useful:

Lemma R3. *For all x, we have*

$$h(x) \le \mathrm{e}^{1+[1/12(p+1)]}\frac{x^{p-1}\,\mathrm{e}^{-x}}{\Gamma(p)}\,\mathrm{e}^{\phi(x,p)},$$

where $\phi(x, p) = x + x\log(x) + p\log(p+1) - (p+x)\log(p+1+x)$. *The function* ϕ *is convex in x, and does not exceed* $\gamma = c(1+\log(c) - \log(c+1)) = 1.045316653509\ldots$ *for all* $x \le c(p+1)$, *where* $c = 1.8442990383\ldots$ *is the solution of* $c + c\log(c) - (c+1)\log(c+1) = 0$. *Thus, for* $x \le c(p+1)$,

$$h(x) \le (1+c)\,\mathrm{e}^{1+[1/12(p+1)]}\frac{x^{p-1}\,\mathrm{e}^{-x}}{\Gamma(p)}.$$

Lemma R3 tells us that for small values of x, the Ressel density is roughly bounded by the gamma density with parameter p. Unfortunately, much more is needed before we can begin thinking about an algorithm, because when Lemma R3 is used for $x \le p$, and Lemma R2 for $x > p$, the integral under the bounding curve tends to ∞ as $p \to \infty$. Also, the general uniformly fast algorithm for log-concave densities (Devroye, 1984) is not applicable here since h is not log-concave. Nevertheless, we will get real help from the structural properties of the Ressel densities, or rather, its close approximations $h(x)$. The properties needed further on are collected in Lemma R4:

Lemma R4. Define $\psi(x) = \log(h(x))$ *(where* $h(x)$ *is defined in Lemma R1). Then* ψ *is twice continuously differentiable on* $(0, \infty)$, $\psi''(x) < 0$ *for* $x \le (p^2 - 1)/2$, *and* $\psi'(x) > 0$ *for* $x \le (p^2 - 1)/4$. *Furthermore, for* $x \le t \le (p^2 - 1)/4$, *we have*

$$f(x) \le h(x) \le h(t)\, \mathrm{e}^{(x-t)\psi'(t)}.$$

The integral over $[0, t]$ *of the upper bound with respect to Lebesgue measure is not larger than* $h(t)/\psi'(t)$.

6.2. A generator when $p > 1$.

We can now describe a rejection algorithm for the Ressel distribution. The algorithm has one design parameter $c > 0$, and uses rejection from different dominating curves according to whether $x \le c(p+1)$ or $x > c(p+1)$. The threshold values will be called $t = c(p+1)$. Since for technical reasons, we need $t \le (p^2-1)/4$, it is necessary to restrict c by $c \le (p-1)/4$. For $x \le t \le (p^2 - 1)/4$, we recall the bound of Lemma R4:

$$f(x) \le h(t)\, \mathrm{e}^{(x-t)\psi'(t)}.$$

The integral over $[0, t]$ of the upper bound with respect to Lebesgue measure is not larger than the integral over all $x \le t$, i.e. $q_l \triangleq h(t)/\psi'(t)$. For $x > t$, we employ part B of Theorem R1, after transforming $f(x)$ to $g(z)$ via the transformation $z := (p+1)^2/(2x)$. Note that $x > c(p+1)$ if and only if $z \le (p+1)/(2c)$. The density of the transformed random variable is bounded as follows for such z:

$$g(z) \le \frac{p\mathrm{e}^{(1/c)}}{(1+p)\sqrt{1 - \dfrac{p}{c(p+1)}}} \left(1 - \frac{p}{c(p+1)}\right) \frac{\mathrm{e}^{-z(1-(p/c(p+1)))}}{\sqrt{\pi z\left(1 - \dfrac{p}{c(p+1)}\right)}}$$

valid for $z \le c(p+1)$. The bound shows the integration constant explicitly: the integral over the positive halfline of the upper bound is

$$q_r \triangleq \frac{pe^{(1/c)}}{(1+p)\sqrt{1 - \dfrac{p}{c(p+1)}}}.$$

The algorithm can now be summarized as follows in raw form:

Ressel density generator

[SET-UP]

Choose $c \in (0, (p-1)/4)$, and set $t \leftarrow c(p+1)$. For a choice of c, see Theorem R2 below. Compute $\lambda \leftarrow \psi'(t)$ (see proof of Lemma R4). Compute $q_l = h(t)/\lambda$ (see Lemma R1) and

$$q_r = \frac{pe^{(1/c)}}{(1+p)\sqrt{1 - \dfrac{p}{c(p+1)}}}.$$

[GENERATOR]
REPEAT
Generate a uniform [0, 1] random variate U.

IF $U \leq \dfrac{q_l}{q_l + q_r}$

THEN
Generate an exponential random variate E.
Set $X \leftarrow t - E/\lambda$.
If $X \geq 0$
THEN
Generate a uniform [0, 1] random variate V.
Define Accept $\leftarrow [Vq_l \lambda e^{(X-t)\lambda} \leq f(X)]$ (or, equivalently, Accept $\leftarrow [Vq_l \lambda e^{-E} \leq f(X)]$).
ELSE Set Accept $\leftarrow$ False.
ELSE
Generate a normal random variable N.

Set $Z \leftarrow N^2 \bigg/ \left(2 - \dfrac{2p}{c(p+1)}\right)$.

Set $X \leftarrow (p+1)^2/(2Z)$.

IF $X < t$
THEN Set Accept $\leftarrow$ False.
ELSE
Generate a uniform [0, 1] random variable V.

Define Accept $\leftarrow \left[Vq_r \left(1 - \dfrac{p}{c(p+1)}\right) \dfrac{e^{-N^2/2}}{\sqrt{\pi N^2/2}} \leq \dfrac{X}{Z} f(X) \right]$.

(The last condition can be simplified to Accept

$\leftarrow \left[Vq_r |N| e^{-N^2/2}/\sqrt{2\pi} \leq Xf(X) \right]$.)

UNTIL Accept.
RETURN X.

In the algorithm, the left dominating curve is picked with probability $q_l/(q_l + q_r)$, and the right dominating curve with probability $q_r/(q_l + q_r)$. Our choice of c will guarantee that for large values of p, q_l/q_r is negligible and that the expected number of iterations before halting, $q_l + q_r$, is close to one. It is thus approximately correct that for large p, one Ressel variate is obtained at the expense of about two uniform random variates (U and V) and one normal random variate. For additional savings, the "unused portion" of U can be used to construct V.

The interesting reader will have no difficulty with the details of the algorithm. He should note that we have used the fact that $g(Z) = Xf(X)/Z$. Some computational difficulties may arise in the evaluation of T and $f(X)$. It is recommended to compute $\log(T)$ and $\log(f(X))$ instead. The log-gamma function is typically available in built-in mathematical libraries. Should it not be available, one may be forced to apply the alternating series method based upon a convergent series expansion for the log-gamma function, see e.g. Lemma X.1.2 in Devroye (1986).

Further time savings can result if one makes good use of the squeeze principle; Stirling's formula for the gamma function should provide a good starting point.

This brings us, finally, to the choice of c. We first offer general guidelines in Theorem R2 that are related to the asymptotic behavior of c as a function of p. The fact that $q_l + q_r \to 1$ as $p \to \infty$ means that the algorithm effectively incorporates the limit law of Theorem R1. Within the conditions of Theorem R2, the freedom should be exploited to get good performance (small $q_l + q_r$) for moderate and small values of p. Technically speaking, the algorithm is valid for any $p > 1$. In practice, we recommend the above algorithm only for $p \geq 2$, with the choice of c given in Theorem R2.

Theorem R2. Complexity. If c is chosen as a function of p such that

$$\lim_{p\to\infty} c = \infty; \quad \lim_{p\to\infty} \frac{c}{p} = 0; \quad c \leq \frac{p-1}{4} \quad \text{for all} \quad p > 1,$$

then the expected number of iterations of the algorithm $(q_r + q_l)$ *tends to one as* $p \to \infty$. *Furthermore,* $q_l/q_r \to 0$ *as* $p \to \infty$.

If we choose

$$c = \min\left(\frac{p-1}{4}, \left(p \log^{-1}\left(\frac{8p^2}{9\pi}\right) - \frac{1}{2}\right)_+\right),$$

then $q_l + (q_r - 1) \sim (3 \log p)/p$ *as* $p \to \infty$.

6.3. A generator for small values of the parameter

For $p < 1$, the previous bounding technique is not valid. It can't even be adapted for the situation at hand, since f has an infinite peak at the origin (it varies as x^{p-1} as $x \downarrow 0$). For $x \leq c(p+1)$, where c is as in Lemma R3, we will use the bound of Lemma R3 in terms of a gamma function. For $x \geq c(p+1)$, we can still use the transformation $z := (p+1)^2/(2x)$, and apply, for $z \leq (1+p)/(2c)$,

$$g(z) \leq \frac{p\,\mathrm{e}^{(1/c)}}{(1+p)\sqrt{1 - \dfrac{p}{c(p+1)}}}\left(1 - \frac{p}{c(p+1)}\right)\frac{\mathrm{e}^{-z(1-p/c(p+1))}}{\sqrt{\pi z\left(1 - \dfrac{p}{c(p+1)}\right)}}.$$

The area under the bound of Lemma R3 is $q_l = (1 + c)\exp(1/12(p+1))$, while the area under the rightmost piece is

$$q_r = \frac{p\,\mathrm{e}^{(1/c)}}{(1+p)\sqrt{1-\dfrac{p}{c(p+1)}}}.$$

As $p \downarrow 0$, $q_r \to 0$. Hence the main contribution comes from the left part now, which covers the infinite peak at the origin. It is noteworthy that uniformly over all p, we have

$$q_l + q_r \le (1+c)\,\mathrm{e}^{1/12} + \frac{\mathrm{e}^{(1/c)}}{\sqrt{1-\dfrac{1}{c}}} = 5.6333127710\ldots.$$

However, for $p \ge 2$, and especially for very large p, the algorithm of the previous section is faster. The algorithm given below requires a good gamma generator for all parameter values. For shape parameters less than one, see e.g. Vaduva (1977), Ahrens and Dieter (1974), Best (1983), Johnk (1964), Berman (1971), or Devroye (1986, pp. 419–420). The details are as follows:

Ressel density generator

[SET-UP]
Let $c = 1.8442990383\ldots$ be as in Lemma R3, and set $t \leftarrow c(p+1)$. Compute

$$q_l = (1+c)\,\mathrm{e}^{1/(12(p+1))} \text{ and } q_r = \frac{p\,\mathrm{e}^{(1/c)}}{(1+p)\sqrt{1-\dfrac{p}{c(p+1)}}}.$$

[GENERATOR]
REPEAT
    Generate a uniform [0, 1] random variate U.
    IF $U \le \dfrac{q_l}{q_l + q_r}$
        THEN
            Generate a gamma random variate X with parameter p.
            IF $X \le t$
                THEN
                Generate a uniform [0, 1] random variate V.
                Define Accept $\leftarrow \left[V q_l \dfrac{X^{p-1}\,\mathrm{e}^{-X}}{\Gamma(p)} \le f(X) \right]$.
                ELSE Set Accept $\leftarrow$ False.

ELSE
Generate a normal random variable N.

Set $Z \leftarrow N^2 / \left(2 - \frac{2p}{c(p+1)}\right)$.

Set $X \leftarrow (p+1)^2/(2Z)$.

IF $X < t$
THEN Set Accept ← False.
ELSE
Generate a uniform [0, 1] random variable V.

Define Accept $\leftarrow \left[Vq_r \left(1 - \frac{p}{c(p+1)}\right) \frac{e^{-N^2/2}}{\sqrt{\pi N^2/2}} \le \frac{X}{Z} f(X) \right]$.

(The last condition can be simplified to Accept

$\leftarrow \left[Vq_r |N| \frac{e^{-N^2/2}}{\sqrt{2\pi}} \le Xf(X) \right]$.)

UNTIL Accept.
RETURN X.

7. Appendix

Proof of Lemma P1. We start with

$$r(n) \triangleq \frac{p_{n+1}}{p_n} = \left(1 + \frac{\lambda}{\lambda n + p}\right)^n (\lambda n + p) \frac{e^{-\lambda}}{n+1}.$$

Replace n by a real number x, and note that

$$\begin{aligned} \frac{\partial}{\partial x} \log(r(x)) &= \log\left(\left(1 + \frac{\lambda}{\lambda x + p}\right)\right) + \frac{\lambda^2 + \lambda p}{(\lambda x + p)(\lambda(x+1) + p)} - \frac{1}{x+1} \\ &\le \frac{\lambda - p}{(\lambda x + p)(x+1)} + \frac{\lambda^2 + \lambda p}{(\lambda x + p)(\lambda(x+1) + p)} \\ &= \frac{2\lambda^2(x+1) + \lambda p - p^2}{(x+1)(\lambda x + p)(\lambda(x+1) + p)}, \end{aligned}$$

where we used the inequality $\log(1 + u) \le u$. The numerator in the upper bound is non-positive if $x+1 \le p(p-\lambda)/(2\lambda^2)$. Thus, $r(x)$ is log-concave for such values of x. Furthermore, if we consider the same range on n, we see that

$$\frac{p_n}{p_{n-1}} \ge \frac{p_{n+1}}{p_n}.$$

This concludes the proof of Lemma P1.

Proof of Lemma P3. We recall the definition of $r(x)$ from the proof of Lemma P1. Clearly,

$$\begin{aligned}\log(r(x)) &= x\log\left(1+\frac{\lambda}{\lambda x+p}\right)+\log(\lambda x+p)-\lambda-\log(x+1)\\ &\le \frac{\lambda x}{\lambda x+p}-\lambda+\log\left(\frac{\lambda x+p}{x+1}\right)\\ &= \frac{\lambda x}{\lambda x+p}-\lambda+\log\left(1+\frac{(\lambda-1)x+(p-1)}{x+1}\right)\\ &\le \frac{\lambda x}{\lambda x+p}-\lambda+\frac{(\lambda-1)x+(p-1)}{x+1}\\ &= \frac{x(p\lambda-p-\lambda^2)+(p^2-p-p\lambda)}{(\lambda x+p)(x+1)}.\end{aligned}$$

The right side is ≤ 0 when $x \ge (p^2-p-p\lambda)/(p+\lambda^2-p\lambda)$. This happens for all x when $p \le 1+\lambda$ (in which case the distribution is monotone $\downarrow$). Assume thus that $p > 1+\lambda$. By property P1, unimodality follows if

$$\frac{p^2-p-p\lambda}{p+\lambda^2-p\lambda} \le \frac{p^2-2\lambda^2-p\lambda}{2\lambda^2}.$$

This leads to the inequality

$$p^3(1-\lambda)-p^2\lambda+3p\lambda^3-2\lambda^4 \ge 0,$$

which is satisfied if $p \ge \lambda/(1-\lambda)$. Finally, for $x=(p-1)/(1-\lambda)$, we have

$$\begin{aligned}\log(r(x)) &= \frac{p-1}{1-\lambda}\log\left(1+\frac{\lambda(1-\lambda)}{p-\lambda}\right)+\log\left(\frac{p-\lambda}{1-\lambda}\right)-\lambda-\log\left(\frac{p-\lambda}{1-\lambda}\right)\\ &\le \frac{\lambda^2-\lambda}{p-\lambda}.\end{aligned}$$

This shows that the mode is $\le \lceil (p-1)/(1-\lambda)\rceil$. To obtain a lower bound for the mode, take $d \le p/\lambda$ (later on, it will be set equal to $(1+\lambda^2)/(1-\lambda)$). Then, using the inequality $\log(1+u) \ge u/(1+u)$, valid for $u \ge 0$, we see that

$$\begin{aligned}\log\left(r\left(\frac{p-d}{1-\lambda}\right)\right) &\ge \frac{p-d}{1-\lambda}\,\frac{\lambda(1-\lambda)}{p-\lambda d+\lambda(1-\lambda)}-\lambda+\log\left(\frac{p-\lambda d}{p-d+1-\lambda}\right)\\ &= \frac{(1-\lambda)(-d\lambda-\lambda^2)}{p-\lambda d+\lambda-\lambda^2}-\log\left(1+\frac{1-\lambda-d+\lambda d}{p-\lambda d}\right)\\ &\ge -(1-\lambda)\frac{(\delta\lambda+\lambda^2)(p-\lambda d)+(1-d)(p-\lambda d+\lambda+\lambda^2)}{(p-\lambda d)(p-\lambda d+\lambda-\lambda^2)},\end{aligned}$$

which is ≥ 0 if the numerator is ≤ 0. Now replace the value of d by $(1+\lambda^2)/(1-\lambda)$. The numerator then simplifies to

$$\lambda^3-\lambda+p(1+\lambda^2)-d(1-\lambda))=\lambda^3-\lambda\leq 0.$$

Hence, a mode occurs at a value at least equal to $\lceil (p-d)/(1-\lambda)\rceil$. Note finally that if the requirement that $d\leq p/\lambda$ does not hold, then it is certainly true that the mode occurs to the right of $\lceil (p-d)/(1-\lambda)\rceil$, a nonpositive integer.

The last statement follows from Lemma P1 if we can show that

$$\left\lceil\frac{p-1}{1-\lambda}\right\rceil\leq\frac{p^2-p\lambda-2\lambda^2}{2\lambda^2}.$$

This is satisfied if $p^2(1-\lambda)-p(\lambda+\lambda^2)-2\lambda^2+4\lambda^3\geq 0$. The left-hand-side, being quadratic in p, is nonnegative except possibly between its two roots (if they exist). The rightmost of these roots occurs exactly at

$$\frac{\lambda}{2(1-\lambda)}\left(1+\lambda+\sqrt{17\lambda^2-22\lambda+9}\right)\leq\frac{\lambda}{2(1-\lambda)}(1+\lambda+3-\lambda)=\frac{2\lambda}{1-\lambda}.$$

This concludes the proof of Lemma P3.

Proof of Lemma P6. Start from Lemma P5. We use the inequality $\log(1+u)\leq u$ (valid for all $u>-1$) to bound $\log(1+((\lambda-1)n+p-1)/(n+1))$. This gives with a little work the first inequality. The second inequality is obtained by noting that

$$\begin{aligned}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}&=\frac{1}{\sqrt{n+1}}\left(\sqrt{1+\frac{1}{n}}-1\right)\\&\geq\frac{1}{\sqrt{n+1}}\left(\frac{1/n}{2\sqrt{1+1/n}}\right)=\frac{1}{2(n+1)\sqrt{n}}\geq\frac{1}{2(n+1)^{3/2}}.\end{aligned}$$

Proof of Theorem C1. The value of $E(N)$ is immediate from the form of the upper bound. For $p\in[0,1]$, this is maximal when $\lambda=0$, and for $1\leq p\leq 1+\lambda$, it is maximal at $\lambda=p-1$. In the former case, we obtain $E(N)=e^{-p}+pe^2\sqrt{2/\pi}$, and in the latter $E(N)=e^{-p}+pe^{4-2p}\sqrt{2/\pi}$. In both cases, the maximum occurs at $p=1$ and is $1/e+e^2\sqrt{2/\pi}$.

We require some standard bounds for the logarithm that are obtainable from Taylor's series expansion with remainder.

Lemma C1. (Bounds for the logarithm.) For all $u>-1$, we have

$$\log(1+u)\leq u-\frac{u^2}{2(1+u_+)},$$

and for all $u\in(0,1)$,

$$\log(1-u)\geq\max\left(-\frac{u}{(1-u)},\ -u-\frac{u^2}{2(1-u)}\right).$$

Proof of Lemma C2. From Lemma P5, we have

$$p_n \le \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\,\mathrm{e}^{-v+(n-1)\log(1+v/(n+1))}$$

$$= \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\,\mathrm{e}^{-(n+1)(v/(n+1)-\log(1+v/(n+1)))-2\log(1+v/(n+1))}, \tag{2}$$

where $v = x(\lambda-1)+(p-1) = (x+1)(\lambda-1)+(p-\lambda)$. For $v/(n+1) \in (-1, 0]$, we note that the exponent in the upper bound is bounded from above by

$$-\frac{n+1}{2}\left(\frac{v}{n+1}\right)^2 - 2\log\left(1+\frac{v}{n+1}\right) = -\frac{v^2}{2(n+1)} - 2\log\left(1+\frac{v}{n+1}\right).$$

When $v/(n+1) \ge -\epsilon$, this is further bounded from above by

$$\frac{-v^2}{2(n+1)} - 2\log(1-\epsilon).$$

Next, for $v \ge 0$, from (2),

$$p_n \le \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\,\mathrm{e}^{-[(n+1)/2](v/(n+1))^2[1/1+(v/(n+1))]}$$

$$= \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\,\mathrm{e}^{-(v^2/2(n+1+v))}.$$

For $v/(n+1) \le \delta$, we see that the exponent does not exceed $-v^2/(2(n+1)(1+\delta))$. In conclusion, for all $v/(n+1) \in [-\epsilon, \delta]$, we see that

$$p_n \le \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}(1-\epsilon)^2}\,\mathrm{e}^{-[1/(1+\delta)][v^2/2(n+1)]}.$$

For $n \le x \le n+1$, we have $(n+1)^{-3/2} \le x^{-3/2}$. Furthermore, if $\mu = (\mathrm{p}-\lambda)/(1-\lambda)$, we have

$$\frac{v^2}{2(n+1)} - \frac{(1-\lambda)^2}{2}\frac{(x-\mu)^2}{x} = \frac{(1-\lambda)^2}{2}\left(\frac{(n+1-\mu)^2}{n+1} - \frac{(x-\mu)^2}{x}\right)$$

$$= \frac{(1-\lambda)^2}{2}\left(\frac{(n+1-\mu)^2}{n+1} - \frac{(x-\mu)^2}{x}\right)$$

$$= \frac{(1-\lambda)^2}{2}(n+1-x)\left(1-\frac{\mu^2}{(n+1)x}\right)$$

$$\ge \frac{(1-\lambda)^2}{2}\min\left(0, \left(1-\frac{\mu^2}{n(n+1)}\right)\right).$$

The last lower bound is minimal when $n+1$ is replaced by $(p-\lambda)/(1-\lambda+\delta)$.

Doing so shows that

$$\frac{v^2}{2(n+1)} - \frac{(1-\lambda)^2}{2}\frac{(x-\mu)^2}{x}$$
$$\geq -\frac{p\delta(2+\delta-2\lambda)+(1+\delta)(1-\lambda)^2-\lambda(1-\lambda+\delta)^2}{2(p-1-\delta)} = -\phi.$$

Thus, for $n < x \leq n+1$, $v/(n+1) \in [-\epsilon, \delta]$, i.e. for $(p-\lambda)/(1-\lambda+\delta) \leq n+1 \leq (p-\lambda)/(1-\lambda-\epsilon)$, where $0<\epsilon<1-\lambda$, $\delta>0$,

$$p_n \leq \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}(1-\epsilon)^2}\ \mathrm{e}^{-(1-\lambda)^2/2(1+\delta)[(x-\mu)^2/x]}\ \mathrm{e}^{\phi/(1+\delta)}.$$

Now, replace $n+1$ by its lower bound, and x in the denominator of the exponent by the upper bound for $n+1$.

Proof of Lemma C3. We let v be as in the proof of Lemma C2. Consider first the case $v/(n+1) \leq -\epsilon$, noting that for any n, $v/(n+1) \geq -(1-\lambda)$. From (2),

$$p_n \leq \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\ \mathrm{e}^{-(n-1)(v/(n+1)-\log(1+v/(n+1)))-(2v/(n+1))}$$
$$\leq \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\ \mathrm{e}^{-((n-1)v^2/2(n+1)^2)-(2v/(n+1))}$$
$$\leq \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\ \mathrm{e}^{[(n-1)\epsilon v/2(n+1)]+2(1-\lambda)}$$
$$\leq \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}}\ \mathrm{e}^{(1-(2(1-\lambda-\epsilon)/(p-\lambda)))(\epsilon v/2)+2(1-\lambda)},$$

where we used the fact that $(n-1)/(n+1)$ is at least equal to $(u-1)/(u+1)$, evaluated at $u+1=(p-\lambda)/(1-\lambda-\epsilon)$. The first inequality of Lemma C3 is obtained after noting that for $n \leq x \leq n+1$, $v \leq p-\lambda-(1-\lambda)x$.

Consider next the case $n+1 \leq (p-\lambda)/(1-\lambda+\delta)$. Using $n+1 \geq 1$, $v/(n+1+v) \geq \delta/(1+\delta)$, and $n \leq x \leq n+1$, we conclude from a bound used in the proof of Lemma C2,

$$p_n \leq \frac{p}{\sqrt{2\pi}\,(n+1)^{3/2}} e^{-(\delta/2(1+\delta))v} \leq \frac{p}{\sqrt{2\pi}} e^{(\delta(1-\lambda)/2(1+\delta))(x+1-((p-\lambda)/(1-\lambda)))}.$$

Proof of Theorem C2. Consider first the asymptotic optimality property A. Take δ and ϵ such that for each fixed $\lambda \in [0, 1)$, as $p \uparrow \infty$, $\delta \to 0$, $\delta\sqrt{p/\log p} \to \infty$, $\epsilon \to 0$, and $\epsilon\sqrt{p} \to \infty$. Then, we easily verify that $\phi \to 0$, $G \to 1$, $H_r \to 0$ and $H_l \to 0$, as required.

This brings us to the uniform boundedness question for $G+H_l+H_r$. Our proof provides a rather loose uniform upper bound. The choices suggested for ϵ and δ are convenient for our purposes, but they are by no means optimal. We begin by noting that $\epsilon < 1-\lambda$, $\epsilon < 2^{-1/3}$, $\delta < 1 < p-1$, $\delta \leq (1-\lambda)^{2/5}/2^{1/3}$,

$\delta/(1+\delta) < 1/2$, $p/(p-\lambda) \leq 3/2$, $(p-\lambda)(1-\lambda) \geq \max(\lambda, (3-\lambda)(1-\lambda)) = (5-\sqrt{13})/2$, $2+(p-\lambda)(1-\lambda) \leq (p-\lambda)(1-\lambda)\beta^3$, and $\delta \geq (1-\lambda)^{1/15}/(\beta(p-\lambda)^{1/3})$, where $\beta = (1+4/(5-\sqrt{13}))^{1/3}$.

From this, it is easy to see that

$$\frac{\phi}{1+\delta} \leq \frac{2\delta p + (1-\lambda)^2}{2p-4} \leq \frac{2p+1}{2p-4} \leq \frac{7}{2}.$$

Also,

$$G \leq \frac{p}{p-\lambda}\, \mathrm{e}^{\phi/(1+\delta)} \frac{\sqrt{1+\delta}}{(1-\epsilon)^2} \leq \frac{3}{2}\, \mathrm{e}^{7/2}\sqrt{2}\,(1-2^{-1/3})^{-2}.$$

To bound H_r, we observe that $t_r - \mu \geq 0$, and that $1 - 2(1-\lambda-\epsilon)/(p-\lambda) \geq 1 - 2(1-\lambda)/(p-\lambda) \geq 1/3$. Thus,

$$H_r \leq \frac{2p\sqrt{1-\lambda}\ \mathrm{e}^{2(1-\lambda)}(2+(1-\lambda)(p-\lambda))^{1/3}}{\sqrt{2\pi}\,(p-\lambda)^{3/2}(1-\lambda)/3}$$

$$\leq \frac{9\,\mathrm{e}^2(2+(1-\lambda)(p-\lambda))^{1/3}}{\sqrt{2\pi}\sqrt{(p-\lambda)(1-\lambda)}} \leq \frac{9\,\mathrm{e}^2(2+(5-\sqrt{13})/2)^{1/3}}{\sqrt{2\pi}\sqrt{(5-\sqrt{13})/2}}.$$

Finally, when bounding H_l, we need the fact that $t_l + 1 - \mu \leqslant 1 - \delta(p-\lambda)/(1-\lambda)(1-\lambda+\delta)$. Thus, we have

$$H_l \leq \frac{4p}{\sqrt{2\pi}\,\delta(1-\lambda)}\, e^{(1-[\delta(p-\lambda)/(1-\lambda)(1-\lambda+\delta)])(\delta(1-\lambda)/2(1+\delta))}$$

$$\leq \frac{4p\ \mathrm{e}^{1/4}}{\sqrt{2\pi}\,\delta(1-\lambda)}\ \mathrm{e}^{-(\delta^2(p-\lambda)/2(1+\delta)(1-\lambda+\delta))}$$

$$\leq \frac{4p\ \mathrm{e}^{1/4}\beta(p-\lambda)^{1/3}}{\sqrt{2\pi}\,(1-\lambda)^{16/15}}\ \mathrm{e}^{-((1-\lambda)^{2/15}(p-\lambda)^{1/3}/4\beta^2(1-\lambda)^{2/5}(1+2^{-1/3}))}$$

$$\leq \frac{4p^{4/3}\ \mathrm{e}^{1/4}\beta}{\sqrt{2\pi}\,(1-\lambda)^{16/15}}\ \mathrm{e}^{-((2/3)^{1/3}p^{1/3}/4\beta^2(1-\lambda)^{4/15}(1+2^{-1/3}))}.$$

If we set $u = p^{5/4}/(1-\lambda)$, then the last upper bound is of the general form (with positive constants A, B),

$$Bu^{16/15}\ \mathrm{e}^{-Au^{4/15}} \leq B\left(\frac{4}{Ae}\right)^4.$$

Proof of Lemma C4. The polynomial bound is obtained in Lemma P6. From Lemma P5, we recall the definition of ρ_u. Thus,

$$\rho_u' = \log\left(\frac{\lambda u + p}{u+1}\right) + 1 - \lambda + \frac{1/2}{u+1} - \frac{\lambda + p}{\lambda u + p}$$

and

$$\rho_u'' = \frac{\lambda}{\lambda u + p} - \frac{1}{u+1} - \frac{1/2}{(u+1)^2} + \frac{\lambda^2 + \lambda p}{(\lambda u + p)^2}$$

$$= \frac{u^2(3\lambda^2/2) + u(\lambda p + 3\lambda^2 - p^2) + (\lambda^2 + 2\lambda p - 3p^2/2)}{(\lambda u + p)^2 (u+1)^2}.$$

But $p \geq 1 + \lambda$ implies that $p \geq 2\lambda^2/(3-\lambda)$ and thus that $\lambda^2 + 2\lambda p - 3p^2/2 \leq 0$. Hence, $\rho_0'' \leq 0$. Examining the numerator of ρ_u'' we thus see that ρ_u'' is nonpositive on some interval $[0, u^*]$, and nonnegative on $[u^*, \infty)$ for some positive number u^*. In particular, we claim that u^* is at least as large as

$$v \triangleq \frac{2}{3} \frac{p^2 - \lambda p - 3\lambda^2}{3\lambda^2}.$$

Indeed, this follows from the fact that the rightmost root of a quadratic equation $x^2 + ax + b = 0$ is at least equal to $-a$ when $b \leq 0$. In our case, we have to verify whether $\lambda^2 + 2\lambda p - 3p^2/2 \leq 0$. This requires that $p \geq (\lambda/3)(2 + \sqrt{10})$. This condition in turn is satisfied since $p \geq 1 + \lambda$. We also have to verify that the solution itself, v, is nonegative, to avoid trivialities. This leads to the condition that $p \geq (1/2)(\lambda + \sqrt{12\lambda + \lambda^2})$ (a sufficient condition for this is that $p \geq 3$). So, we have shown that $\rho_v'' \leq 0$, that ρ_u is unimodal, and that ρ_u' is $\downarrow$ on $[0, v]$. Finally, by Taylor's series expansion with remainder,

$$\log p_n \leq \rho_n \leq \rho_v + (n - v)\rho_v'$$

for all $n \leq v$. The same is true if v is replaced by u with $u < v$ in the last inequality.

Proof of Theorem C3. The condition on α implies that $\alpha < 1/(1 + 2c/3)$, and thus that the limit given in the statement of the theorem is a finite positive number. We will use the fact that $E(N) = q_l + q_r$. It is easy to see that $v \sim 2p^2/3$ as $p \to \infty$. Hence,

$$\lim_{p \to \infty} q_r = \sqrt{\frac{3}{\pi\alpha}}.$$

We will see that $\rho_t' \to 0$ as $p \to \infty$, so that the limit of q_l is equal to the limit of e^{ρ_t}/ρ_t'. Furthermore, this limit remains the same if the integer t is replaced by the real number $u = \alpha v$. Using the fact that $\log(1 + x) = x - x^2/2 + O(x^3)$ as $x \to 0$, we see that

$$\rho_u = \log p + 1 - p - \tfrac{1}{2}\log(2\pi) + (u-1)\log\left(1 + \frac{(\lambda - 1)u + p - 1}{u+1}\right)$$

$$- \tfrac{3}{2}\log(u+1) + u(1-\lambda)$$

$$= \log p + 1 - p - \tfrac{1}{2}(2\pi) + \frac{u-1}{u+1}((\lambda - 1)u + p - 1)$$
$$- \frac{u-1}{2}\left(\frac{(\lambda-1)u+p-1}{u+1}\right)^2$$
$$- \tfrac{3}{2}\log(2\alpha p^2/3) + u(1-\lambda) + o(1)$$
$$= -2\log p - \tfrac{1}{2}\log(2\pi) - \tfrac{3}{2}\log(2\alpha/3) - \frac{3}{4\alpha}\left(1 - \frac{2\alpha c}{3}\right)^2 + o(1).$$

Thus,

$$\lim_{p\to\infty} p^2\, e^{\rho_v} = \frac{1}{\sqrt{2\pi}\,(2\alpha/3)^{3/2}}\, e^{-(1-2\alpha c/3)^2/(4\alpha/3)}.$$

Furthermore,

$$p^2\rho_u' = \frac{-(\lambda-1)+(p-1)+1/2}{u+1}p^2 - \frac{p^2}{2}\left(\frac{(\lambda-1)u+p-1}{u+1}\right)^2$$
$$+ O\left(\frac{1}{p}\right) - \frac{\lambda+p}{\lambda u+p}p^2$$
$$= \frac{p^2(p^2 - up(1-\lambda) - \lambda - \lambda^2 u - \lambda u/2 - 3p/2)}{(u+1)(\lambda u+p)}$$
$$- \frac{1}{2}\left(\frac{1-2\alpha c/3}{2\alpha/3}\right)^2 + o(1)$$
$$= \frac{1-\alpha-2\alpha c/3}{(2\alpha/3)^2} - \frac{1}{2}\left(\frac{1-2\alpha c/3}{2\alpha/3}\right)^2 + o(1).$$

Combining all this shows that

$$\frac{e^{\rho_x}}{\rho_u'} \to \frac{\sqrt{2\alpha/3}}{\sqrt{2\pi}}\,\frac{e^{-(1-2\alpha c/3)^2/(4\alpha/3)}}{1-\alpha-2\alpha c/3-\frac{1}{2}(1-2\alpha c/3)^2}.$$

This concludes the proof of the convergence result. The limit for $E(N)$ is minimal for the a function of α with a unique minimum. The last statement of the Theorem is left as an exercise.

Proof of Lemma R2. Use the inequality $\log(1+u) \ge u - u^2/2$, valid for all $u > 0$ (but only recommended for $1 \ge u \ge 0$) to obtain the bound

$$\left(1 + \frac{p+1}{x}\right)^{-(p+x)} \le e^{-(p(p+1)/x)-(p+1)+((p+1)^2 p/2x^2)+((p+1)^2/2x)}.$$

Plug this into the definition of $h(x)$ in Lemma R1.

Proof of Theorem R1. The inequality at the top of the Theorem follows from Lemma R2, if we use standard methods for computing densities of transformed random variables; in this case, $z := (p+1)^2/(2x)$ and $dz := -(p+1)^2/(2x^2)\,dx$ are the formal transformations required.

Statement A follows from the fact that the upper bound for $g(z)$ tends to the density $e^{-z}/\sqrt{\pi z}$ pointwise; hence $g(z)$ must do so pointwise. Furthermore, by Scheffe's lemma (1947), pointwise convergence of densities implies convergence in total variation.

For statement B, we argue as follows:

$$g(z) \le \frac{p}{1+p}\, e^{2c+(2pzc/(p+1))} \frac{e^{-z}}{\sqrt{\pi z}}$$

$$\le \frac{p\, e^{2c}}{(1+p)\sqrt{1-\dfrac{2pc}{p+1}}}\left(1-\frac{2pc}{p+1}\right)\frac{e^{-z(1-[2pc]/[p+1])}}{\sqrt{\pi z\left(1-\dfrac{2pc}{p+1}\right)}}.$$

Proof of Lemma R3. We first use the inequality $\log(1+u) \ge u/(1+u)$ (valid for all $u > 0$) to obtain

$$\left(1+\frac{x+1}{p}\right)^{p+x} \ge e^{(p+x)((x+1)/(x+1+p))} \ge e^{-x},$$

for all x and p. By Lemma R1, the fact that $\Gamma(p+1) = p\Gamma(p)$, and Stirling's approximation,

$$h(x) = \frac{p\, e^{p+1}}{\sqrt{2\pi}}\left(\frac{x}{p+1}\right)^{p+x}\left(1+\frac{x}{p+1}\right)^{-(p+x)}\frac{1}{x\sqrt{1+p+x}}$$

$$\le \Gamma(p+1)\frac{\sqrt{p+1}\; e^{p+1}}{(p+1)^{p+1}\sqrt{2\pi}}\,\frac{x^{p-1}\, e^{-x}}{p\Gamma(p)}\, x^x\, e^x (p+1)^{p+1}\left(\frac{1}{p+1}\right)^{p+x}$$

$$\times\left(1+\frac{x}{p+1}\right)^{-(p+x)}\frac{p}{\sqrt{p+1}\sqrt{1+p+x}}$$

$$\le e^{(1/12(p+1))}\frac{x^{p-1}\, e^{-x}}{\Gamma(p)}\left(\frac{x}{p+1}\right)^{x} e^x\left(1+\frac{x}{p+1}\right)^{-(p+x)}$$

$$= e^{(1/12(p+1))}\frac{x^{p-1}\, e^{-x}}{\Gamma(p)}\, e^{\phi(x,p)}.$$

The derivative of ϕ is $1+\log(x)-\log(p+1+x)+1/(p+1+x)$, and the second derivative is $(px+(p+1)^2)/(x(p+1+x)^2)$, which is always positive. Hence, on any interval, ϕ attains its maximum at one of the two ends. At $x=0$, we have $\phi(x, p) = 0$. For $x \le c(p+1)$ with c variable for the time being, we see that

$$\phi(x, p) \le \max(0,\ p(c+\log(c)-(c+1)\log(c+1)) + c(1+\log(c)-\log(c+1))).$$

It is convenient to choose c such that the coefficient of p is zero. This yields the value of c given in the statement of the Lemma. For that value, and all $x \le c(p+1)$, we have

$$\phi(x, p) \le c(1 + \log(c) - \log(c+1)) = \log(c+1).$$

This concludes the proof of Lemma R3.

Proof of Lemma R4. We start with $\psi(x) = \log(p) + (p+1) - (1/2)\sqrt{2\pi} + (p + x - 1)\log(x) - (p + x + (1/2))\log(p + x + 1)$, and observe that

$$\psi'(x) = \log(x) - \log(p+x+1) + \frac{p+x-1}{x} - \frac{p+x+1/2}{p+x+1}$$
$$= \log\left(\frac{x}{p+x+1}\right) + \frac{p-1}{x} + \frac{1/2}{p+x+1}.$$

Also,

$$\psi''(x) = \frac{1}{x} - \frac{1}{p+x+1} - \frac{p-1}{x^2} - \frac{1/2}{(p+x+1)^2}$$
$$= \frac{2x - p^2 + 1}{x^2(p+x+1)} - \frac{1/2}{(p+x+1)^2} < 0,$$

for $2x \le p^2 - 1$. Thus, ψ' is nonincreasing as $x \uparrow$. We only verify that it is positive when $4x \le p^2 - 1$. Using the inequality $\log(1+u) \le 2u/(2+u)$, valid for all $u > 0$, we note that, temporarily setting $x = c(p+1)$,

$$\psi'(c(p+1)) = \log\left(\frac{c}{c+1}\right) + \frac{p-1}{c(p+1)} + \frac{1/2}{(p+1)(c+1)}$$
$$= \frac{1}{c} - \log\left(1 + \frac{1}{c}\right) - \frac{2}{c(p+1)} + \frac{1/2}{(p+1)(c+1)}$$
$$\ge \frac{1}{c(2c+1)} - \frac{2}{c(p+1)} + \frac{1/2}{(p+1)(c+1)}$$
$$\ge \frac{1/2}{(p+1)(c+1)} > 0$$

when $2c + 1 \le (p+1)/2$. This is satisfied if and only if $x \le (p^2 - 1)/4$.

The second part of the Lemma follows directly from Taylor's series expansion with remainder.

Proof of Theorem R2. Without work, we have $q_r \to 1$ as $p \to \infty$ under the stated conditions. We will now show that $q_l \to 0$ as $p \to \infty$. We first recall that

$$\psi'(t) \ge \frac{1}{c(2c+1)} - \frac{2}{c(p+1)} + \frac{1/2}{(p+1)(c+1)} \sim \frac{1}{c(2c+1)}$$

since $c = o(p)$. Also, using $1 - (1+c)\log(1 + 1/c) \le -1/(2c+1)$, we have

$$h(t) \le \frac{p}{\sqrt{2\pi}} \frac{\sqrt{(c+1)(p+1)}}{(c(p+1))^2} e^{-(p+1)/(2c+1)} \sim \frac{1}{\sqrt{2\pi p}} c^{-3/2} e^{-p/(2c+1)},$$

because $c \to \infty$. Thus,

$$q_l = \frac{h(t)}{\psi'(t)} \le (1+o(1))\sqrt{\frac{2c}{\pi p}}\ \mathrm{e}^{-(p/(2c+1))} \to 0 \quad \text{as} \quad p \to \infty.$$

To choose c, we minimize $q_l + (q_r - 1)$. Let us indicate very roughly how one can proceed. First, under the conditions of Theorem R2, $q_r - 1 \sim 3/2c$. Equating this with the asymptotic bound for q_l given above, we obtain a close-to-optimal c. One functional iteration started at $c = p$ yields the value

$$c = \frac{p}{\log\left(\frac{8p^2}{9\pi}\right)} - \frac{1}{2}.$$

With this choice, $q_l + q_r - 1 \sim q_l \sim (3 \log p)/p$ as $p \to \infty$.

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