A NOTE ON FINDING CONVEX HULLS VIA MAXIMAL VECTORS

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1. Introduction

The problem of finding the convex hull of n points has received widespread attention in the past decade. In particular, if $X_1, ..., X_n$ are independent identically distributed random vectors from R^d with common density f, the following questions were investigated: if C is the complexity of the convex hull algorithm for $X_1, ..., X_n$ (thus, C is a random variable), then how do ess sup C (the 'worst-case complexity') and E(C) (the 'average complexity') increase with n for particular densities f?

There are algorithms that have worst-case complexity O(n log n) for all densities f [1,5,10,11] on R^2. The algorithms of Jarvis [6] and Eddy [4] have worst-case complexity $O(n^2)$.

Recently, several algorithms were shown to exhibit linear average complexities (E(C) = O(n)) for certain classes of densities on R^2:

(1) The 'divide and conquer' method of Bentley and Shamos does so whenever E(N), the expected number of points on the convex hull, satisfies $E(N) = O(n^p), p < 1$. The latter condition is fulfilled, for example, when f is the uniform density on a convex r-gon [9] or when f is normal [8].

(2) The elimination method of Toussaint [11] is known to do so for uniform densities on the unit square, and for all radial densities with a monotone and slow-varying tail [3].

(3) The recent method of Bentley et al. [2] that is based upon first finding the set of maximal vectors, has E(C) = O(n) whenever f can be written as a d-fold product of densities:

$$f(x_1, ..., x_d) = \prod_{i=1}^{d} f_i(x_i) . \qquad (1)$$

This is true, e.g., for the normal density.

The purpose of this paper is to show (3) and to obtain a few additional results on the distribution of M, the number of maximal vectors.

We say that a vector x_1 is maximal among $(x_1, ..., x_n)$ when none of the other vectors dominates it in every component. In other words, the positive quadrant centered at x_1 has no other point in it. In fact, one can define for each quadrant, $1 \leq i \leq 2^d$:

M(i) = number of maximal vectors for i^{th} quadrant among $X_1, ..., X_n$.

It is clear that when f satisfies (1), the average number of maximal vectors taken from all quadrants does not exceed

$$E(\Sigma_i M(i)) = 2^d E(M) \qquad (2)$$

and

$$E((\Sigma_i M(i))^p) \leq E(\Sigma_i 2^{d(p-1)} M^p(i)) = 2^{dp} E(M^p), \quad p \geq 1 . \qquad (3)$$

In (2) and (3) we use M for M(1), the number of maximal vectors in the first quadrant.

By Theorem 3 of [2] we can find all the maximal vectors among $X_1, ..., X_n$ in R^d in expected time O(n) when f satisfies (1). If one uses a convex hull algorithm with worst-case complexity $O(n^p)$, $p \geq 1$, on the set of all maximal vectors (there are at most $\Sigma_i M(i)$ of them), then the overall average complexity of the convex hull procedure is

$$E(C) \leq k_1 n + k_2 E\{M^p\}, \quad (4)$$

where k_1, k_2 are constants possibly depending upon p and d, but not on n.

Here we show the following:

Theorem 1. If (1) holds, then, for every $p > 1$, there exists $g(p) > 0$ with

$$E(M^p) \leq g(p)(E(M))^p, \quad (5)$$

where $g(p) = (1 + \lceil p \rceil^2 + \cdots + \lceil p \rceil^{\lceil p \rceil})^{p/\lceil p \rceil}$ and $\lceil \cdot \rceil$ is the ceiling function.

Of course, by Jensen's inequality, it is always true that

$$E(M^p) \geq (E(M))^p, \quad (6)$$

and this, together with (5) shows the closeness of $E(M^p)$ to $(E(M))^p$.

We also show

Theorem 2. If (1) holds, then

$$\frac{E(M)}{\frac{(\log(n))^{d-1}}{(d-1)!}} \xrightarrow{n} 1. \quad (7)$$

More precisely,

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{\log \log n}{\log n}\right)\frac{(\log n)^{d-1}}{(d-1)!} \leq$$
$$\leq E(M) \leq \sum_{i=0}^{d-1} \frac{(\log n)^i}{i!}$$
$$\leq \frac{(\log n)^{d-1}}{(d-1)!} + e(\log n)^{d-2}. \quad (8)$$

The proof of Theorem 2 is entirely probability theoretical, and the result (7) is not obtainable from the combinatorial inequalities of Bentley [2]. From (4), (5) and (7) we have without calculation:

Theorem 3. If (1) holds, then the algorithm which uses the method of [2] to find the maximal vectors, and then uses a worst-case $O(n^p)$ ($p \geq 1$) algorithm to find the convex hull among these points, has average complexity $O(n)$.

Note. In Theorem 3, p and d are arbitrary. Actually, it is known that a worst-case $O(n^{d+1})$ algorithm always exists for any d.

2. Proofs

In view of (1), we can and do assume that $X_1, \ldots, X_n$ are independent and uniformly distributed in $[0, 1]^d$. Also, we will write $X_i = (X_{i1}, \ldots, X_{id})$ when we need the individual components of X_i.

Clearly,

$$\begin{aligned} E(M) &= n P(X_1 \text{ is a maximal vector}) \\ &= n E((1 - (1 - X_{11}) \cdots (1 - X_{1d}))^{n-1}) \\ &= n E((1 - X_{11} X_{12} \cdots X_{1d})^{n-1}) \\ &= E\left(\frac{1}{X_{12} \cdots X_{1d}}(1 - (1 - X_{12} \cdots X_{1d})^n)\right), \end{aligned} \quad (9)$$

where we have used the integral $\int_0^1 (1 - za)^{n-1}\, dz = (1 - (1 - a)^n)/na$ with $a = X_{12} \cdots X_{1d}$, $z = X_{11}$. From (9) it is clear that E(M) increases with n.

Proof of Theorem 1. We show Theorem 1 for n even and $p = 2$. The other cases follow trivially. Let M' be the number of maximal vectors among $X_1, \ldots, X_{n/2}$. Then, if I is the indicator function,

$$\begin{aligned} E(M^2) &= E((\Sigma_i I_{[X_i \text{ is a maximal vector}]})^2) \\ &= \Sigma_{i,j} P(X_i \text{ is a maximal vector}, \\ &\qquad X_j \text{ is a maximal vector}) \\ &= \Sigma_i P(X_i \text{ is a maximal vector}) \\ &\quad + n(n-1) P(X_1 \text{ and } X_2 \text{ are maximal vectors}) \\ &= E(M) + n(n-1) P(X_1 \text{ and } X_2 \text{ are maximal vectors}) \\ &\leq (E(M))^2 + n P(X_1 \text{ maximal vector among } X_1, X_3, X_5, \ldots) \\ &\quad \times (n-1) P(X_2 \text{ maximal vector among } X_2, X_4, \ldots) \end{aligned}$$

$$\leqslant (E(M))^2 + (2\,E(M'))^2$$
$$\leqslant (E(M))^2 + 4(E(M))^2$$
$$= 5\,(E(M))^2 .$$

For p integer, the proof is analogous. Let M' be the number of maximal vectors among $X_1, ..., X_{n/p}$, where we assume that n is a multiple of p. It is easy to obtain the inequality

$$E(M^p) \leqslant E(M) + (p\,E(M'))^2 + (p\,E(M'))^3 + \cdots + (p\,E(M'))^p \leqslant (E(M))^p\,(1 + p^2 + p^3 + \cdots + p^p) .$$

For p not integer, we have

$$E(M^p) \leqslant (E(M^{\lceil p \rceil}))^{p/\lceil p \rceil} \leqslant [(E(M))^{\lceil p \rceil}(1 + \lceil p \rceil^2 + \lceil p \rceil^3 + \cdots + \lceil p \rceil^{\lceil p \rceil})]^{p/\lceil p \rceil} = (E(M))^p\,g(p) .$$

Proof of Theorem 2. The density of $Y = X_{12} \cdots X_{1d}$ is

$$h(y) = \frac{1}{(d-2)!}\left(\log\frac{1}{y}\right)^{d-2}, \quad 0 < y < 1 . \qquad (10)$$

To see this, use the facts that $-\log X_{12}$ is exponentially distributed, that the sum of $(d-1)$ independent exponential random variables is gamma $(d-1)$, and proceed as follows:

$$P\{Y \leqslant y\} = P\{-\log X_{12} - \log X_{13} - \cdots - \log X_{1d} \geqslant -\log y\} = \int_{-\log y}^{\infty} \frac{u^{d-2}}{(d-2)!}\,e^{-u}\,du .$$

Next, use the transformation $u = \log z$, $du = (-1/z)\,dz$.

With (10), we can rewrite (9) as

$$E(M) = \int_0^1 \frac{1}{y}\left(\log\frac{1}{y}\right)^{d-2} \frac{1}{(d-2)!}\,(1 - (1-y)^n)\,dy . \qquad (11)$$

To find an upper-bound for this, we have

$$E(M) \leqslant \int_{1/n}^{1} \frac{1}{y}\left(\log\frac{1}{y}\right)^{d-2} \frac{1}{(d-2)!}\,dy + n\int_0^{1/n}\left(\log\frac{1}{y}\right)^{d-2}\frac{1}{(d-2)!}\,dy$$

because $(1-y)^n \geqslant 1 - ny$. Since, by partial integration,

$$(d-1)\int_\beta^1 \frac{1}{p}\left(\log\frac{1}{y}\right)^{d-2} dy = \left(\log\frac{1}{\beta}\right)^{d-1},$$

the first of these terms is equal to $(\log n)^{d-1}/(d-1)!$ The second one is equal to

$1 + \log n/1! + \cdots + (\log n)^{d-2}/(d-2)!$

in view of the recursive relation

$$\int_0^{1/n}\left(\log\frac{1}{y}\right)^{d-2}\frac{1}{(d-2)!}\,dy = \frac{1}{n}(\log n)^{d-2}\frac{1}{(d-2)!} + \int_0^{1/n}\left(\log\frac{1}{y}\right)^{d-3}\frac{1}{(d-3)!}\,dy, \quad d \geqslant 1 .$$

Therefore,

$$E(M) \leqslant \sum_{i=0}^{d-1}\frac{(\log n)^i}{i!} \leqslant \frac{(\log n)^{d-1}}{(d-1)!} + e(\log n)^{d-2} . \qquad (12)$$

Furthermore,

$$E(M) \geqslant \int_{\alpha/n}^{1}(1 - e^{-\alpha})\frac{1}{y}\left(\log\frac{1}{y}\right)^{d-2}\frac{1}{(d-2)!}\,dy = \frac{(1-e^{-\alpha})\left(\log\frac{n}{\alpha}\right)^{d-1}}{(d-1)!} \qquad (13)$$

for arbitrary $\alpha \in (0, n)$. Picking $\alpha = \log n$ shows that

$$\frac{E(M)}{\frac{(\log n)^{d-1}}{(d-1)!}} \geqslant \left(1 - \frac{1}{n}\right)\left(1 - \frac{\log\log n}{\log n}\right)^{d-1} \xrightarrow{n} 1 .$$

Also, from (12),

$$\frac{E(M)}{\frac{(\log n)^{d-1}}{(d-1)!}} \leqslant 1 + \frac{e(d-1)!}{\log n} \xrightarrow{n} 1 ,$$

concluding the proof of Theorem 2.

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