On the Convergence of Statistical Search

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Abstract—The convergence of statistical (random) search for the minimization of an arbitrary multimodal functional $Q(w)$ is dealt with by using the theorems of convergence of random processes of Braverman and Rozonoer. It is shown that random search can be regarded as a gradient algorithm in the Q-domain. Using this gradient to define the minimum of the functional, the convergence to this minimum is discussed at length. The theorems proved in this paper apply as well to discrete as to continuous optimization problems and as such, the developed technique is competitive with stochastic automata with a variable structure. The optimality of the scheme follows from the convergence in probability of the average performance to the minimum. The freedom in the organization of the search within the boundaries outlined by the conditions of convergence is emphasized. Finally, it is pointed out how various mixed random search and hierarchical search systems fall into the domain of application of the theorems.

Introduction

THE PROBLEM of optimizing a multimodal and unknown functional $Q(w)$ with respect to a set of m parameters w is solved through direct search techniques. Usually, random search [6], [7], [11], [12], [16], [18], [19], [27] or stochastic automata with a variable structure [1], [4], [8], [17] are used when the functional (performance index) is extremely complicated; e.g., $Q(w)$ has many discontinuities, has many local minima, is very nonlinearly shaped, is not differentiable, etc. If more information about the functional is available (usually in terms of differentiability, smoothness, etc.), combinations of the aforementioned methods with local hill-climbing methods can be considered [4], [5], [15], [16], [23], [24]. Most of these local hill-climbing methods (for a recent survey, see [26]) are derived from the Kiefer–Wolfowitz stochastic approximation algorithm [14], [25] and stochastic gradient algorithms [2], [3], [13], [21].

While the convergence of most of the hill-climbing algorithms has been investigated at length in the literature, there are only a few papers that deal with the convergence of random search, mainly because of the mathematical intractability of the algorithm and the difficulty of dealing with multimodal functionals [6], [7], [18], [19]. Most of the research in the field of random search has been concerned with the organizational aspect of the search [13], [15], [16], [19], [23], [24], and hence heuristic techniques are legion. In comparison, this paper is a unified presentation dealing with different modes of convergence (convergence with probability one, in probability, of the mean) for different quantities (the sequence of estimates of the minimum, the average performance, etc.). The strongest result obtained until now was the convergence in probability to its minimal value for an accumulated average performance [18].

The mathematical difficulties are overcome here by projecting the whole problem in the $Q(w)$ domain. It is shown that there exists a unimodal convex function in this domain with respect to which the random search algorithm can be regarded as a gradient algorithm. The convergence is proved by using the theorems of convergence of random processes in machine learning [9]. Old results [6], [7], [18], [19] are strengthened and generalized at the same time. The class of functionals to be allowed includes "anomalous functionals" (an anomalous functional is such that the value of $Q(w)$ does not convey any information as to the value of any other point $w^* \neq w$). In order to allow such general performance indices, a new definition of the minimum has to be introduced. The stationary points of the convex function derived in this paper uniquely define the minimal value of the performance index.

The random search algorithm of Matyas [19] and Gurin [6] is generalized in order to obtain a gradual reduction of the overall average performance. It is emphasized that within the bounds dictated by the conditions of convergence, the organization of the search depends only upon the imagination of the designer. This freedom can be used to develop convergent nonheuristic hierarchical search systems, mixed random and nonrandom search systems, search systems with subsequent linear search in special directions (as in deterministic optimization), etc.

Problem Formulation

Let $w \in W \subseteq E^m$, m-dimensional Euclidean space. $Q(w)$ is the expected value of a random variable ζ/w conditioned on w. The pdf of ζ/w is $\pi(\zeta/w)$:

$$Q(w) \triangleq E\{\zeta/w\} = \int \zeta\pi(\zeta/w)\,d\zeta. \qquad (1)$$

ζ will be referred to as a measurement of the performance index $Q(w)$. $Q(w)$ is sometimes called a noisy surface or stochastic performance index. If $\pi(\zeta/w) = \delta(\zeta - Q(w))$, $Q(w)$ is said to be deterministic. In analogy with the terminology used in automata theory, $\pi(\zeta/w)$ defines a random environment. The environment is a P-model environment if $\zeta \in \{0,1\}$ and an S-model environment if $\zeta \in [0,1]$. In general, the environment is called a Q-model environment.

The presented procedure is iterative with iteration counter j. The "state" of the system (minimization procedure) is denoted by $X_j \in X$. X is the state space. Essential in the class of probabilistic procedures discussed in this paper is the concept of basepoint w_j (best estimate of the minimum up to the jth iteration) which is included in X_j. The tech-

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nique consists of a suitable rule for updating w_j and X_j in such a way that convergence is guaranteed. $\mathscr{X} = \{X_j\}_{j \geq 0}$ is called the state sequence. X_j, which might contain parameters to be learned or adapted during the search, is a random element defined on some (usually growing) probability space. Functions of the state that map X into R^1 are thus random variables on this probability space.

The procedure is defined as follows.

1) X_0 (and thus w_0) is given or selected in X. The initial distribution of w_0 in W is denoted by $g_B(w)$.

2) w_j is applied to the environment; $\lambda_{Bj} \geq 1$ iid (independent identically distributed) measurements ζ/w_j are observed (measured), totalled, and averaged. The average is denoted by ζ_j.

3) A trial point w^*_{j+1} is generated. With probability at least equal to $\alpha_j \in [0,1]$, w^*_{j+1} is purely randomly generated according to a given fixed pdf $g_B(w)$ concentrated on W. With the complementary probability, w^*_{j+1} is the outcome of some computing procedure or process which is not necessarily a random vector-generating process; for instance, any direct search procedure and any degree of human intervention in the search process are allowed. It is supposed that w^*_{j+1} has, in the latter case, a pdf $g^*(w/X_j)$ which is completely determined by the knowledge of X_j. The pdf of w^*_{j+1} conditioned on X_j equals:

$$g(w/X_j) = \alpha_j g_B(w) + (1 - \alpha_j) g^*(w/X_j). \tag{2}$$

The number of auxiliary measurements needed in this stage is denoted by λ_{Gj}.

4) The trial point w^*_{j+1} is applied to the environment; $\lambda_{Tj} \geq 1$ iid measurements ζ/w^*_{j+1} are observed, totalled, and averaged. The average is denoted by ζ^*_{j+1}.

5) The new state X_{j+1} is computed. This includes the adaptation of the basepoint by the decision rule:

$$w_{j+1} = \begin{cases} w^*_{j+1} & \text{if } \zeta_j^* < \zeta_j - \varepsilon_j \\ w_j & \text{otherwise} \end{cases} \tag{3}$$

where $\varepsilon_j \geq 0$ is some positive threshold.

6) Operations 2)—5) constitute one basic cycle or "iteration" of the search process.

A flow chart of a generalized random search algorithm is shown in Fig. 1.

Remark 1: $\{\lambda_{Bj}\}$, $\{\lambda_{Gj}\}$, $\{\lambda_{Tj}\}$, $\{\varepsilon_j\}$, $\{\alpha_j\}$ are treated in the following as number sequences. Notice however that adaptive parameter sequences can be allowed if there are lower and upper adaptation boundaries that are number sequences satisfying the conditions of convergence to be established in this paper.

Remark 2: In the jth cycle, the number of measurements equals $\lambda_{Bj} + \lambda_{Tj} + \lambda_{Gj}$. The properties of convergence, however, are with respect to j, while the cost of optimization is often proportional to the number of performance index evaluations.

Remark 3: $g_B(w)$ is the basic random law of generation of test points and defines W. If $g_B(w)$ is concentrated on a finite set of points in E^m, then W is a finite set, and the minimization problem is reduced to a strategy selection problem. If a minimum is to be found in an n-dimensional

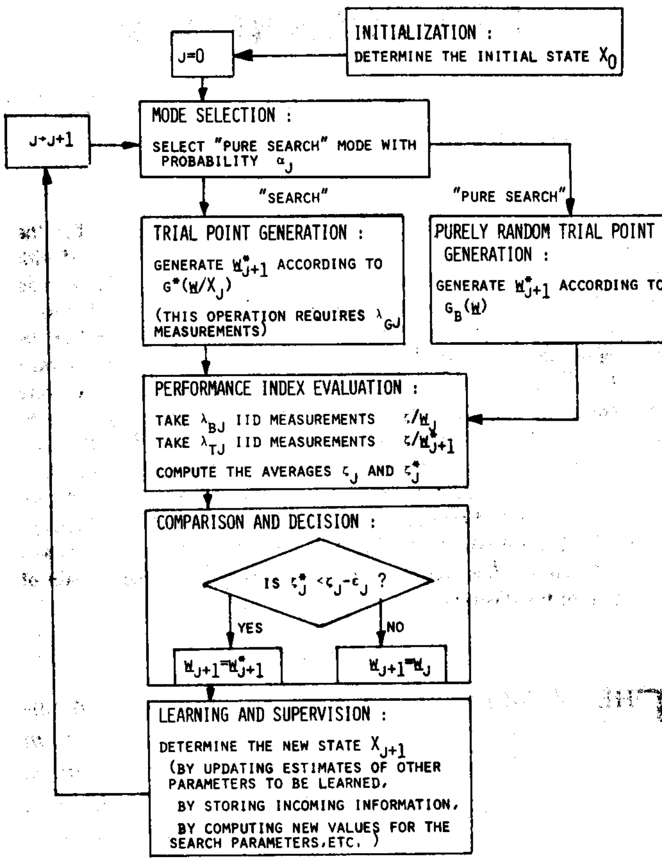

Fig. 1. Flow chart of generalized random search algorithm.

hypercube, $g_B(w)$ could be the uniform pdf in this hypercube. If W is unbounded but at the same time there are strong indications that the minimum is close to w^0, then $g_B(w)$ might be centered at w^0 and have a Gaussian form with a fixed positive definite covariance matrix. Thus $g_B(w)$ reflects the *a priori* knowledge about the problem and is considered to be given.

Remark 4: It is assumed that $w^*_{j+1} \in W$, for all $X_j \in X$.

Remark 5: The nature of $g^*(w/X_j)$ and X_j is of no major importance for establishing the convergence of the algorithm. Hence, to increase the efficiency of the procedure, techniques from direct deterministic optimization theory (linear search in a special direction, creeping random search, partan techniques, etc. [29]), hierarchical search systems, all kinds of "almost"-gradient stochastic minimization algorithms [2], [3], [12], [21], [25], [26] and combined global search and stochastic approximation methods [4], [5], [15], [24], [30] can be incorporated in the search scheme. Notice that due to the option that α_j might tend to zero, $g(w/X_j)$ might tend to $g^*(w/X_j)$ on which no restrictions are imposed. The purely random way of generating trial points will only be needed in the beginning to scan the complete search domain.

Remark 6: Gurin's scheme [6] is obtained for $\lambda_{Tj} = \lambda_{Bj} = \lambda_j$, $\lambda_{Gj} = 0$, $\varepsilon_j = \varepsilon > 0$.

Remark 7: Constraints are easily dealt with in random search [29] and cause such minor trouble that they are not explicitly discussed here. The most popular ways of dealing with constraints are the rejection method (reject all the trial points that fall outside the permitted region W) and the variable-weight Lagrange multiplier method [31], [32].

Notations and Definitions

The problem will now be translated into the q-domain. Notice that every pdf in W induces a corresponding pdf in the q-domain; $p_B(q)$ defined by (4) will be called the *basic search pdf* and $p(q/X_j)$ induced by $g(w/X_j)$ will be referred to as the *search pdf:*

$$p_B(q) \triangleq \int_W \delta(q - Q(w))g_B(w)\, dw \tag{4}$$

$$p(q/X_j) \triangleq \int_W \delta(q - Q(w))g(w/X_j)\, dw. \tag{5}$$

The corresponding cdf's (cumulative distribution functions) are $P_B(q)$ and $P(q/X_j)$ for which:

$$P_B(q) = \int_{-\infty}^{q]} p_B(u)\, du = \int_W \omega(Q(w),q)g_B(w)\, dw$$
$$= \text{Prob}\,\{Q(w_{j+1}^*) \leq q/\alpha_j = 1\} \tag{6}$$

$$P(q/X_j) = \int_{-\infty}^{q]} p(u/X_j)\, du = \int_W \omega(Q(w),q)g(w/X_j)\, dw$$
$$= \text{Prob}\,\{Q(w_{j+1}^*) \leq q/X_j\} \tag{7}$$

where $\omega(a,b) \triangleq 1$ for $a \leq b$, $\omega(a,b) \triangleq 0$ for $a > b$. An important inequality independent of X_j is (8), which stems from combining (2) with (6)–(7):

$$P(q/X_j) \geq \alpha_j \cdot P_B(q) \qquad p(q/X_j) \geq \alpha_j \cdot p_B(q). \tag{8}$$

Thus $P(q/X_j)$ is the distribution function of $Q(w_{j+1}^*)$ conditioned on X_j.

State functions of some importance in classical optimization theory include the value of the performance index at the basepoint (9) and the indicator of the event $\{Q(w_j) \leq q_{\min} + \eta\}$, where $\eta > 0$ is some constant and $q_{\min}$ is the minimum of $Q(w)$ which will be defined later in a rigorous way:

$$q_j \triangleq Q(w_j) \tag{9}$$

$$\omega(q_j,q_{\min} + \eta) \triangleq \text{ind}\,\{Q(w_j) \leq q_{\min} + \eta\}. \tag{10}$$

In automata theory and in several applications it is not only important that q_j tends to $q_{\min}$ as $j \to \infty$ but also that the average measured performance tends to this value. Therefore, define the *mean of the search pdf* as

$$\rho(X_j) \triangleq \int q \cdot p(q/X_j)\, dq = E\{Q(w_{j+1}^*)/X_j\} \tag{11}$$

and the mean over all the expected values of the auxiliary measurements made in the w_{j+1}^* generating process if $\alpha_j = 0$ as $\rho_G(X_j)$. The *average measured performance* $\psi(X_j)$ is then defined by

$$\psi(X_j) \triangleq \frac{\lambda_{Bj} \cdot q_j + \lambda_{Tj} \cdot \rho(X_j) + \lambda_{Gj} \cdot \rho_G(X_j)}{\lambda_{Bj} + \lambda_{Tj} + \lambda_{Gj}}. \tag{12}$$

The *degree of concentration* $D(X_j,\eta)$ on $(-\infty,q_{\min} + \eta]$ is defined by

$$D(X_j,\eta) \triangleq \frac{\lambda_{Bj} \cdot \omega(q_j,q_{\min} + \eta) + \lambda_{Tj} \cdot P(q_{\min} + \eta/X_j) + \lambda_{Gj} \cdot P_G(q_{\min} + \eta/X_j)}{\lambda_{Bj} + \lambda_{Tj} + \lambda_{Gj}} \tag{13}$$

where $P_G(q_{\min} + \eta/X_j)$ denotes the probability that if one of the λ_{Gj} auxiliary measurements is picked at random, the value of the performance index at that point is not greater than $q_{\min} + \eta$. $D(X_j,\eta)$ denotes the probability that if one of the $\lambda_{Bj} + \lambda_{Tj} + \lambda_{Gj}$ measurements made at the jth cycle is picked at random, the corresponding expected value of the performance index at the point of measurement is less than or equal to $q_{\min} + \eta$.

Of theoretical interest will be the integral

$$\int_{-\infty}^{q_j]} P_B(u)\, du \tag{14}$$

to be studied later. Finally, introduce the bar operator $\bar{a}$ with the following meaning: $\bar{a} = \max\,\{a,q_{\min}\}$. Then, before investigating the properties of convergence of the cited state functions, $q_{\min}$ will be defined.

A New Definition of the Minimum

Consider the positive goal function $I(q)$ and assume that $\lim_{q\to-\infty} q \cdot P_B(q) = 0$:

$$I(q) \triangleq \int_{-\infty}^{q]} \int_{-\infty}^{x]} (q - u) \cdot p_B(u)\, du\, dx$$
$$= \int_{-\infty}^{q]} \int_{-\infty}^{x]} P_B(u)\, du\, dx. \tag{15}$$

This goal function is convex (the second derivative is $P_B(q) \geq 0$), nonnegative and continuous. Its gradient is given by

$$I'(q) = \int_{-\infty}^{q]} P_B(u)\, du = \int_{-\infty}^{q]} (q - u) \cdot p_B(u)\, du$$
$$= P_B(q) \cdot \left[q - \int_{-\infty}^{q]} u \cdot \frac{p_B(u)}{P_B(q)}\, du\right]. \tag{16}$$

The gradient is always positive except if either $P_B(q) = 0$ or the mean over all $Q(w) \leq q$ equals q when w has pdf $g_B(w)$.

The *minimal value* (minimum) of $Q(w)$ with respect to $g_B(w)$ is $q_{\min}$: $q_{\min} = \inf\,\{q \in R \mid I'(q) > 0\}$. Trivially, $q_{\min}$ satisfies:

$$\int_{-\infty}^{q_{\min}]} P_B(u)\, du = 0. \tag{17}$$

To each $\eta > 0$, there exists a number $\delta(\eta) > 0$ such that

$$\int_{-\infty}^{q_{min}+\eta]} P_B(u)\, du \geq \delta(\eta) > 0. \qquad (18)$$

The minimum is such that the set $\{w \in W: Q(w) < q_{min}\}$ has a zero measure with respect to $g_B(w)$. This follows directly from definition (17) and property (4). It can also be shown that the given definition is in accordance with most of the definitions encountered in the literature for continuous or piecewise continuous functions.

Notice that it is not necessarily true that $P_B(q_{min}) = 0$. One can easily imagine a staircase function $Q(w)$ with a set of points having a nonzero measure with respect to $g_B(w)$ and for which $Q(w) = q_{min}$. Clearly, $P_B(q_{min}) > 0$. It is not hard to see that

$$q_{min} = \inf \{q \in R \mid P_B(q) > 0\} \qquad (19)$$

and therefore, to each $\eta > 0$, there exists a number $\theta(\eta) > 0$ such that

$$P_B(q_{min} + \eta) \geq \theta(\eta) > 0. \qquad (20)$$

Convergence of the State Functions

Assign the following letters to denote the convergence of the corresponding state function:

A	convergence of $\bar{q}_j$	to q_{min}
B	convergence of $\omega(q_j,q_{min} + \eta)$	to 1, for every $\eta > 0$
C	convergence of $D(X_j,\eta)$	to 1, for every $\eta > 0$
D	convergence of $\int_{-\infty}^{q]} P_B(u)\, du$	to 0
E	convergence of $\bar{\psi}(X_j)$	to q_{min}.

A second index added to it concerns the mode of convergence:

1 convergence with probability one
2 convergence in probability
3 convergence in the mean.

Finally, introduce the conditions:

$$|Q(w)| \leq q_{max} < \infty, \qquad \text{for all } w \in W \qquad (21)$$

$$\lim_{j\to\infty} \frac{\lambda_{Gj} + \lambda_{Tj}}{\lambda_{Bj} + \lambda_{Gj} + \lambda_{Tj}} = 0. \qquad (22)$$

Then the following relations exist between the properties of convergence of the given random variables.

1) $Y1 \rightarrow Y2$, $(Y = A,B,C,D,E)$ since convergence with probability one implies convergence in probability.

2) $Y3 \rightarrow Y2$, $(Y = A,B,C,D,E)$ since convergence in the mean implies convergence in probability.

3) $Y2 \rightarrow Y3$, $(Y = B,C)$: $1 \geq D(X_j,\eta) \geq 0$. Then $E\{D(X_j,\eta)\} \geq (1 - \eta')\cdot \text{Prob}\, \{D(X_j,\eta) \geq 1 - \eta'\}$ with $\eta' > 0$. Since $D(X_j,\eta) \rightarrow 1$ in probability, $E\{D(X_j,\eta)\} \rightarrow 1$ also. Convergence in the mean follows since one is the upper limit of $D(X_j,\eta)$. The proof for $\omega(q_j,q_{min} + \eta)$ is similar.

4) (21) $\oplus$ $Y2 \rightarrow Y3$, $Y = A,D,E$: The proof is similar if it is noted that all the random variables converge to their lower limit and are deterministically bounded from above.

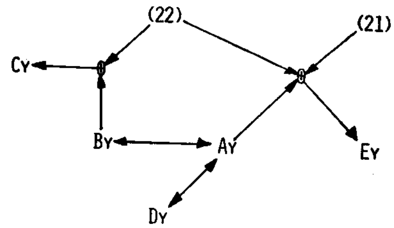

Fig. 2. Relations between properties of convergence of different state functions for $y = 1,2$.

Indeed, $\bar{\psi}(X_j) \leq q_{max}$; $\bar{q}_j \leq q_{max}$ and integral (14) is never greater than $(q_{max} - q_{min})$.

5) (21) $\oplus$ (22) $\oplus$ $Ay \rightarrow Ey$, $y = 1,2,3$:

$$q_{min} \leq \bar{\psi}(X_j) \leq \frac{\lambda_{Bj}\cdot \bar{q}_j + (\lambda_{Tj} + \lambda_{Gj})\cdot q_{max}}{\lambda_{Tj} + \lambda_{Gj} + \lambda_{Bj}} \rightarrow \bar{q}_j, \qquad \text{for } j \rightarrow \infty$$

by using definition (12), (21), and (22). The rest of the proof is trivial.

6) (22) $\oplus$ $By \rightarrow Cy$, $y = 1,2,3$: The proof is as under 5) if it is noted that

$$1 \geq D(X_j,\eta) \geq \frac{\lambda_{Bj}\cdot \omega(q_j,q_{min} + \eta) + 0\cdot(\lambda_{Tj} + \lambda_{Gj})}{\lambda_{Bj} + \lambda_{Tj} + \lambda_{Gj}} \rightarrow \omega(q_j,q_{min} + \eta), \qquad \text{for } j \rightarrow \infty.$$

7) $Ay \leftrightarrow By$, $y = 1,2$. Indeed, take $y = 1$ for instance. Then from Ay:

$$\lim_{j\to\infty} \text{Prob}\, \{\max_{i\geq j} \bar{q}_i \leq q_{min} + \eta\} = \lim_{j\to\infty} \text{Prob}\, \{\min_{i\geq j} \omega(q_i,q_{min} + \eta) = 1\} = 1$$

which is exactly the definition of $B1$, since this holds for every $\eta > 0$ by supposition.

8) $Ay \leftrightarrow Dy$, $y = 1,2$. Indeed, note that

$$\int_{-\infty}^{q_j]} P_B(u)\, du = \int_{-\infty}^{\bar{q}_j]} P_B(u)\, du$$

is a continuous function of $\bar{q}_j - q_{min}$ only vanishing for $\bar{q}_j - q_{min} = 0$. The same applies for $\bar{q}_j - q_{min}$ and thus, applying lemma 2 of [22], the convergence either with probability one or in probability of one of both random variables to zero implies the convergence in the same mode of the other random variable to zero.

The relations between the properties of convergence of the different state functions for $y = 1,2$, are shown in Fig. 2.

Using these properties it is seen that the convergence of

$$\bar{q}_j, \int_{-\infty}^{q_j]} P_B(u)\, du$$

or $\omega(q_j,q_{min} + \eta)$, for all $\eta > 0$, is crucial. Therefore, we will focus our attention on the convergence of the latter state functions.

Theorems of Convergence of Statistical Search

The decision process (3) can mathematically be formulated in terms of the random variable $\xi(w_j,w_{j+1}^*)$ given X_j:

$$\xi(w_j,w_{j+1}^*) \triangleq \begin{cases} 1, & \text{if } w_{j+1} = w_{j+1}^* \text{ (and thus } q_{j+1} = Q(w_{j+1}^*) = u) \\ 0, & \text{if } w_{j+1} = w_j \text{ (and thus } q_{j+1} = q_j). \end{cases} \quad (23)$$

Notice that

$$E\{\xi(w_j,w_{j+1}^*)\} = \text{Prob}\,\{w_{j+1} = w_{j+1}^*/w_j,w_{j+1}^*,X_j\}$$

does depend upon $\lambda_{Bj},\lambda_{Tj}$ and ε_j. Thus, given X_j:

$$w_{j+1} = \begin{cases} w_j \text{ with probability } 1 - \int_W E\{\xi(w_j,w^*)\} \cdot g(w^*/X_j)\, dw^* \\ w^* \text{ with probability } E\{\xi(w_j,w^*)\} \cdot g(w^*/X_j). \end{cases} \quad (24)$$

The pdf $f(w/X_j)$ of w_{j+1} conditioned on X_j is given by

$$E\{\xi(w_j,w)\} \cdot g(w/X_j) + \left(1 - \int_W E\{\xi(w_j,w^*)\} \cdot g(w^*/X_j)\, dw^*\right) \cdot \delta(w - w_j).$$

Define now the random variable $\xi^*(q_j,u) \in \{0,1\}$ such that

$$E\{\xi^*(q_j,u)\} = \begin{cases} \inf\limits_{\substack{\{w \in W: Q(w)=Q(w_j)=q_j\} \\ \{w^* \in W: Q(w^*)=Q(w_{j+1}^*)=u\}}} E\{\xi(w,w^*)\}, & \text{if } q_j \geq u \\ \sup\limits_{\substack{\{w \in W: Q(w)=Q(w_j)=q_j\} \\ \{w^* \in W: Q(w^*)=Q(w_{j+1}^*)=u\}}} E\{\xi(w,w^*)\}, & \text{if } q_j < u. \end{cases} \quad (25)$$

Clearly,

$$E\{\xi^*(q_j,u)\} \geq \text{Prob}\,\{w_{j+1} = w_{j+1}^*/w_j,w_{j+1}^*,q_j \geq u,X_j\}$$
$$E\{\xi^*(q_j,u)\} \leq \text{Prob}\,\{w_{j+1} = w_{j+1}^*/w_j,w_{j+1}^*,q_j < u,X_j\}.$$

Thus

$$\bar{q}_{j+1} = \begin{cases} \bar{q}_j \begin{cases} \text{with probability smaller than } 1 - \int E\{\xi^*(q_j,u)\} \cdot p(u/X_j)\, du, & \text{if } u \leq q_j \\ \text{with probability greater than } 1 - \int E\{\xi^*(q_j,u)\} \cdot p(u/X_j)\, du, & \text{if } u > q_j \end{cases} \\ \bar{u} \begin{cases} \text{with probability greater than } E\{\xi^*(q_j,u)\} \cdot p(u/X_j), & \text{if } u \leq q_j \\ \text{with probability smaller than } E\{\xi^*(q_j,u)\} \cdot p(u/X_j), & \text{if } u > q_j. \end{cases} \end{cases} \quad (26)$$

Thus

$$\{\bar{q}_{j+1} - q_{\min}/X_j\} \leq (\bar{q}_j - q_{\min}) - \int_{-\infty}^{+\infty} E\{\xi^*(q_j,u)\} \cdot (\bar{q}_j - \bar{u}) \cdot p(u/X_j)\, du. \quad (27)$$

After introducing the step function:

$$\phi(q_j,u) \triangleq \begin{cases} 1, & \text{if } u \leq q_j \\ 0, & \text{if } u > q_j \end{cases} \quad (28)$$

(27) can be rewritten as

$$\begin{aligned} E\{\bar{q}_{j+1} - q_{\min}/X_j\} &\leq (\bar{q}_j - q_{\min}) - \int_{-\infty}^{+\infty} \phi(q_j,u) \cdot p(u/X_j) \cdot (\bar{q}_j - \bar{u})\, du \\ &\quad + \int_{-\infty}^{+\infty} (\bar{q}_j - \bar{u}) \cdot E\{\phi(q_j,u) - \xi^*(q_j,u)\} \cdot p(u/X_j)\, du. \end{aligned} \quad (29)$$

The third term on the right side in (29) can be considered as positive noise and disappears only if $\xi^*(q_j,u) \equiv \phi(q_j,u)$, for all q_j and u, which is only the case for deterministic performance indices $Q(w)$ if $\varepsilon_j = 0$. Furthermore,

$$\begin{aligned} \int_{-\infty}^{+\infty} \phi(q_j,u) \cdot p(u/X_j) \cdot (\bar{q}_j - \bar{u})\, du &= \int_{-\infty}^{q_{\min}]} (\bar{q}_j - \bar{u}) \cdot p(u/X_j)\, du + \int_{(q_{\min},}^{q_j]} (\bar{q}_j - \bar{u}) \cdot p(u/X_j)\, du \\ &= (\bar{q}_j - q_{\min}) \cdot P(q_{\min}/X_j) + \int_{(q_{\min},}^{q_j]} (q_j - u) \cdot p(u/X_j)\, du \\ &= (\bar{q}_j - q_{\min}) \cdot P(q_{\min}/X_j) + \int_{(q_{\min},}^{q_j]} P(u/X_j)\, du \end{aligned} \quad (30)$$

which, using inequality (8) is not less than

$$\alpha_j \cdot \left[(\bar{q}_j - q_{\min}) \cdot P_B(q_{\min}) + \int_{-\infty}^{q_j]} P_B(u)\, du\right] \quad (31)$$

since

$$\int_{-\infty}^{q_{\min}]} P_B(u)\, du = 0.$$

Furthermore,

for $q_j \geq u$: $\bar{q}_j - \bar{u} \leq q_j - u$ and $E\{\phi(q_j,u) - \xi^*(q_j,u)\} \geq 0$

for $q_j < u$: $\bar{q}_j - \bar{u} \geq q_j - u$ and $E\{\phi(q_j,u) - \xi^*(q_j,u)\} \leq 0$.

Together, if the second term in (29) is replaced by (31) and if in the third term $(\bar{q}_j - \bar{u})$ is replaced by $(q_j - u)$, the following stronger inequality is obtained:

$$\begin{aligned} E\{U(X_{j+1})/X_j\} &\leq U(X_j) - \alpha_j \cdot V(X_j) + Z(X_j) \\ U(X_j) &= \bar{q}_j - q_{\min} \\ V(X_j) &= (\bar{q}_j - q_{\min}) \cdot P_B(q_{\min}) + \int_{-\infty}^{q_j]} P_B(u)\, du \\ Z(X_j) &= \int_{-\infty}^{+\infty} (q_j - u) \cdot E\{\phi(q_j,u) - \xi^*(q_j,u)\} \cdot p(u/X_j)\, du. \end{aligned} \quad (32)$$

The functions U, V, and Z are all nonnegative, and hence, the inequality (32) is of the type studied by Braverman and Rozonoer [9]. Note that both $U(X_j)$ and $V(X_j)$ are continuous functions of $\bar{q}_j - q_{\min}$ and only vanish for $\bar{q}_j = q_{\min}$. Thus, if $U \to 0$, $V \to 0$ and vice versa. Then, if $E\{U(X_0)\}$ and $E\{V(X_0)\}$ exist,

$$\alpha_j \geq 0 \qquad \sum \alpha_j = \infty \quad , \quad \sum \sup_{X_j \in X} Z(X_j) < \infty \tag{33}$$

there holds:

$$U(X_j) \xrightarrow{\text{with probability one}} 0 \qquad V(X_j) \xrightarrow{\text{with probability one}} 0, \qquad \text{as } j \to \infty. \tag{34}$$

Theorem 1, proved in the Appendix, reduces to a mere check of the condition on $Z(X_j)$.

Define μ_k:

$$\mu_k \triangleq \sup_{w \in W} \{E\{|\zeta - Q(w)|^k/w\}\}$$
$$= \sup_{w \in W} \left\{\int |\zeta - Q(w)|^k \cdot \pi(\zeta/w)\, d\zeta\right\}, \qquad k > 0. \tag{35}$$

Theorem 1: Let there be satisfied (H1)–(H5)

$$E\{|\bar{q}_0|\} < \infty \tag{H1}$$
$$|q_{\min}| < \infty \tag{H2}$$
$$\alpha_j \in [0,1] \qquad \sum_{j=1}^{\infty} \alpha_j = \infty \tag{H3}$$
$$\varepsilon_j \geq 0 \qquad \sum_{j=1}^{\infty} \varepsilon_j < \infty \tag{H4}$$
$$\lambda_{Bj} \geq 1 \qquad \lambda_{Tj} \geq 1 \qquad \lambda_{Gj} \geq 0 \tag{H5}$$

and let the environment be either *deterministic:*

$$\mu_2 = 0 \tag{H6}$$

or *Q-model random* with (H7) and (H8) or (H9) holding:

$$\mu_k < \infty, \qquad k \geq 2, k \text{ even} \tag{H7}$$
$$\sum_{j=1}^{\infty} \frac{1}{\varepsilon_j^{k-1} \cdot \lambda_{Bj}^{k/2}} < \infty \qquad \sum_{j=1}^{\infty} \frac{1}{\varepsilon_j^{k-1} \cdot \lambda_{Tj}^{k/2}} < \infty \tag{H8}$$
$$\sum_{j=1}^{\infty} \lambda_{Bj}^{-1/2} < \infty \qquad \sum_{j=1}^{\infty} \lambda_{Tj}^{-1/2} < \infty \tag{H9}$$

or *S-model random* or *P-model random:*

$$\zeta \in [0,1] \ (S\text{-model}) \qquad \zeta \in \{0,1\} \ (P\text{-model}) \tag{H10}$$
$$\sum_{j=1}^{\infty} (\varepsilon_j \cdot \lambda_{Bj})^{-1} \cdot e^{-\lambda_{Bj} \cdot \varepsilon_j^2/8} < \infty$$
$$\sum_{j=1}^{\infty} (\varepsilon_j \cdot \lambda_{Tj})^{-1} \cdot e^{-\lambda_{Tj} \cdot \varepsilon_j^2/8} < \infty \tag{H11}$$

then the state sequence $\mathscr{X}$ generated by the above given procedure is such that

$$\bar{q}_j \xrightarrow{\text{with probability one}} q_{\min} \qquad \text{as } j \to \infty. \tag{36}$$

The proof of convergence based upon the theorems of Braverman and Rozonoer is given in the Appendix.

Remark 1.1: The conditions are satisfied by taking $\lambda_{Bj} = \lambda_{Tj} = 1$, $\varepsilon_j = 0$ for deterministic environments. Since $Q(w_j) = q_j$ is exactly known from previous measurements, it should not be measured any more at time j, and eventually $\lambda_{Bj} = 0$ is allowable, in which case the algorithm of Matyas is obtained [19].

Remark 1.2: For all the random environments λ_{Bj} and λ_{Tj} should diverge for $j \to \infty$ (see also [6], [18]), and the rate of increase depends upon the rate of decrease of ε_j and upon the conditions imposed upon the nature of the environment (H7), (H10). What makes Theorem 1 interesting is the option of making ε_j equal 0, which is completely new for this type of random search algorithm.

Remark 1.3: The divergence of α_j (H3) insures that the whole search domain will be "searched." However, (H3) does not prevent α_j from decreasing to 0 as $j \to \infty$, which makes this scheme more flexible than the schemes of Gurin [6], Cockrell [16], etc.

An alternative way of dealing with the convergence of $\{\bar{q}_j\}$ is through the indicator function (10) $\omega(q_j, q_{\min} + \eta)$, for all $\eta > 0$. In a way similar to the derivation of (27) it can be shown:

$$E\{1 - \omega(q_{j+1}, q_{\min} + \eta)/X_j\}$$
$$\leq 1 - \omega(q_j, q_{\min} + \eta) - \int E\{\xi^*(q_j,u)\} \cdot (\omega(u, q_{\min} + \eta) - \omega(q_j, q_{\min} + \eta)) \cdot p(u/X_j)\, du$$
$$= (1 - \omega(q_j, q_{\min} + \eta)) - \int_{-\infty}^{q_{\min}+\eta]} E\{\xi^*(q_j,u)/q_j > q_{\min} + \eta\} \cdot (1 - \omega(q_j, q_{\min} + \eta)) \cdot p(u/X_j)\, du + \int_{(q_{\min}+\eta}^{\infty} E\{\xi^*(q_j,u)/q_j \leq q_{\min} + \eta\} \cdot \omega(q_j, q_{\min} + \eta) \cdot p(u/X_j)\, du \tag{37}$$

and from (8), reducing the integration interval in the second term gives

$$E\{1 - \omega(q_{j+1}, q_{\min} + \eta)/X_j\}$$
$$\leq (1 - \omega(q_j, q_{\min} + \eta)) \cdot \left(1 - \alpha_j \cdot \int_{-\infty}^{q_{\min}+\eta/2]} E\{\xi^*(q_j,u)/q_j > q_{\min} + \eta\} \cdot p_B(u)\, du\right) + \int_{(q_{\min}+\eta}^{\infty} E\{\xi^*(q_j,u)/q_j \leq q_{\min} + \eta\} \cdot p(u/X_j)\, du. \tag{38}$$

Introducing

$$Z_{1j} \triangleq \inf_{\theta \in (-\infty, q_{\min}+\eta/2]} E\{\xi^*(q_j,\theta)/q_j > q_{\min} + \eta\}$$
$$Z_{2j} \triangleq \sup_{\theta \in (q_{\min}+\eta, \infty)} E\{\xi^*(q_j,\theta)/q_j \leq q_{\min} + \eta\} \tag{39}$$

we have

$$E\{(1 - \omega(q_{j+1}, q_{\min} + \eta))/X_j\} \leq (1 - \omega(q_j, q_{\min} + \eta)) \cdot (1 - \alpha_j \cdot Z_{1j} \cdot P_B(q_{\min} + \eta/2)) + Z_{2j}. \tag{40}$$

In view of property (20), for all $\eta > 0$: $P_B(q_{\min} + \eta/2) > 0$. The theorems of Braverman and Rozonoer applied to (40) insure that

$$\omega(q_j, q_{\min} + \eta) \xrightarrow{\text{with probability one}} 1, \qquad \text{as } j \to \infty$$

if

$$\alpha_j \cdot Z_{1j} \geq 0 \qquad \sum_{j=1}^{\infty} \alpha_j \cdot Z_{1j} = \infty \qquad \sum_{j=1}^{\infty} Z_{2j} < \infty \quad (41)$$

Further,

$$\omega(q_j, q_{\min} + \eta) \xrightarrow{\text{in probability}} 1, \qquad \text{as } j \to \infty$$

if

$$\alpha_j \cdot Z_{1j} \geq 0 \qquad Z_{2j} \geq 0 \qquad \lim_{j\to\infty} \frac{Z_{2j}}{\alpha_j \cdot Z_{1j}} = 0$$

$$\sum_{j=1}^{\infty} \alpha_j \cdot Z_{1j} = \infty. \quad (42)$$

However, as was proved in the foregoing section, the convergence, for all $\eta > 0$, of $\omega(q_j, q_{\min} + \eta)$ to 1 (with probability one, in probability) is equivalent with the convergence (with probability one, in probability) of $\bar{q}_j$ to $q_{\min}$. Theorem 2, also proved in the Appendix, reduces to a mere check of conditions (41) and (42) by finding explicit expressions for Z_{1j} and Z_{2j}.

Theorem 2: Let there be satisfied (H2), (H3), (H5), (H12):

$$\varepsilon_j \geq 0 \qquad \lim_{j\to\infty} \varepsilon_j = 0 \quad \text{(H12)}$$

and let the environment be either *deterministic* (H6) or *Q-model random* with (H7) and (H13) holding:

$$\sum_{j=1}^{\infty} \frac{1}{\varepsilon_j^{\,k} \cdot \lambda_{Bj}^{\,k/2}} < \infty \qquad \sum_{j=1}^{\infty} \frac{1}{\varepsilon_j^{\,k} \cdot \lambda_{Tj}^{\,k/2}} < \infty \quad \text{(H13)}$$

or *S-model random* or *P-model random* (H10) with:

$$\sum_{j=1}^{\infty} e^{-\lambda_{Bj}\cdot\varepsilon_j^2/2} < \infty \qquad \sum_{j=1}^{\infty} e^{-\lambda_{Tj}\cdot\varepsilon_j^2/2} < \infty \quad \text{(H14)}$$

then the state sequence $\mathscr{X}$ generated by the above given procedure is such that

$$\bar{q}_j \xrightarrow{\text{with probability one}} q_{\min}, \qquad \text{as } j \to \infty.$$

If (H13) and (H14) are replaced by (H15) and (H16), respectively,

$$\lim_{j\to\infty} \frac{1}{\alpha_j \cdot \varepsilon_j^{\,k} \cdot \lambda_{Bj}^{\,k/2}} = 0 \qquad \lim_{j\to\infty} \frac{1}{\alpha_j \cdot \varepsilon_j^{\,k} \cdot \lambda_{Tj}^{\,k/2}} = 0 \quad \text{(H15)}$$

$$\lim_{j\to\infty} \alpha_j^{-1} \cdot e^{-\lambda_{Bj}\cdot\varepsilon_j^2/2} = 0 \qquad \lim_{j\to\infty} \alpha_j^{-1} \cdot e^{-\lambda_{Tj}\cdot\varepsilon_j^2/2} = 0, \quad \text{(H16)}$$

then

$$\bar{q}_j \xrightarrow{\text{in probability}} q_{\min}, \qquad \text{as } j \to \infty. \quad (43)$$

Remark 2.1: The conditions of Theorem 1 and Theorem 2 mainly differ in that the strong condition $\sum \varepsilon_j < \infty$ (H4) is replaced by the weaker condition (H12). Since the rate of decrease of ε_j directly influences the conditions imposed on λ_{Bj} and λ_{Tj}, Theorem 2 can, for some environments, be of more value than Theorem 1.

The Accumulated Average Performance

The theorems do not impose upper bounds on the sampling frequencies λ_{Bj}, λ_{Tj}, and λ_{Gj}. In order to extend the theorems to include convergence with respect to the total number of measurements, it is necessary to introduce such upper bounds. Still, the theorems are concerned with the convergence of some state function, let us say $f(X_j)$, such as the average measured performance, etc.

Another more elegant way of dealing with the convergence with respect to the total number of measurements is through accumulated state functions. The idea of proving the convergence for an accumulated average performance was first mentioned by Gurin [6] and later investigated by Saridis *et al.* [18]. Let λ_j denote $\lambda_{Bj} + \lambda_{Tj} + \lambda_{Gj}$. Then the accumulated state function $f_{Aj} = f_A(X_0, X_1, \cdots, X_j)$ is defined as

$$f_{Aj} = f_A(X_0, X_1, \cdots, X_j) \triangleq \frac{\sum_{i=0}^{j} \lambda_i \cdot f(X_i)}{\sum_{i=0}^{j} \lambda_i}. \quad (44)$$

For all the state functions treated in this paper, it is true that

$$f_0 \leq f(X_i) \leq f_{\max} < \infty, \qquad X_i \in X. \quad (45)$$

Assume further that

$$\lim_{j\to\infty} E\{f(X_j)\} = f_0 \quad (46)$$

$$\sum_{i=1}^{\infty} \lambda_i = \infty \quad (47)$$

then

$$\lim_{j\to\infty} E\{f_A(X_0, \cdots, X_j)\} = f_0. \quad (48)$$

The proof can be found in the Appendix. Under the same conditions it can be proved that convergence with probability one (in probability) of $f(X_j)$ to f_0 induces convergence with probability one (in probability) of f_{Aj} to f_0.

Conclusion

The main advantage of random search (statistical search) over other direct search techniques is its general applicability, i.e., there are almost no conditions concerning $Q(w)$ (continuity, unimodality, convexity, etc.).

If the problem is completely transferred to the Q-domain, even W is unspecified. Therefore, random search is the basic building block for any multimodal search routine, and hence, mixed search techniques are legion. Theorems 1 and 2 deal with global convergence of such procedures both in deterministic and stochastic environments. The main purpose was to discuss a procedure that would include most of the well-known random search algorithms as a special case. In fact, the scheme is so flexible that mixed search procedures [4], [5], [16], [28], hierarchical search, etc., are covered by the theorems of convergence. There is so much freedom in organizing the search that it is hard to speak of an "algorithm" but rather of a "framework" for statistical search procedures.

The scheme does not make use of such properties as "continuity" of $Q(w)$ and is of an "overall" nature. Hence, its rate of convergence can be expected to be inferior when compared with local hill-climbing algorithms if $Q(w)$ is smooth and unimodal. Due to the addition of the "mode selector" (Fig. 1) to the random generator, any other procedure can be incorporated in the program, which is important for large computerized optimization routines. Any amount of on-line adaptation or learning can be allowed, and any method from deterministic or stochastic optimization theory can be inserted, all through the trial point generating process (such as the Kiefer–Wolfowitz gradient, stochastic gradient, creeping random search, the Hooke and Jeeves' method, partan, and even human intervention). An additional feature is that due to the random scanning of W, it is very easy to extract much information concerning $Q(w)$ or $\pi(\zeta/w)$.

The following points should receive more attention.

1) The decision stage in the algorithm is based upon a comparison of two independent sequences. A lot of measurements can be saved by allowing for sequential decision (stopping) rules. Although the authors obtained good experimental results with such rules (see also [16]) it is difficult to derive practical inequalities of the kind used in the theorems of convergence.

2) The scheme is too general and should be efficiently adapted for use in specific environments (deterministic or stochastic; anomalous or smooth; finite set W or Euclidean spaces; unimodal or multimodal; P, S or Q-model; etc.). Particularly, the scheme is competitive with SAVS for use in finite set environments. This comparison deserves more attention, both theoretically and experimentally.

3) The search parameters are number sequences which is as big a disadvantage as it is for stochastic approximation, for instance, because the scheme deals with "global" search. Intuitively, it is felt that "optimal" number sequences, if they exist, will strongly depend upon the form of $p_B(q)$ for anomalous $Q(w)$. Because almost all the basic pdf's appearing in real problems are continuous in q (even if $Q(w)$ is not continuous in W!), it is possible to make some hypotheses about $p_B(q)$ and to derive these theoretically important "optimal" number sequences.

4) A completely new way of proving the convergence of random search is presented here. By extending the state space and finding other Lyapunov-type functions of the state, more sophisticated adaptation algorithms can be treated. Algorithms with many "basepoints" seem interesting in view of their application in multimodal function optimization with several parallel local hill-climbing processes.

5) The deterministic random optimization scheme of Matyas [19] is not deduceable as a special case of the presented scheme due to the presence of the condition $\lambda_{Bj} \geq 1$. This suggests that there must exist a statistical search scheme in which, instead of ζ_j, there is another estimate of $Q(w_j)$ which is based upon the mean of some or all of the previous measurements made at the basepoint while λ_{Bj} might be zero. Such a scheme with an "aging" basepoint is under investigation now.

6) The core of the scheme is the averaging process to obtain estimates of $Q(w)$. For some environments however such mean estimates are out of the question (for instance, because the variance of the noise is infinite) or inefficient when compared with other statistics. If the noise satisfies some regularity conditions, quantile estimators (median, etc.) can be used instead. In heavy noise situations, the use of thresholds to keep ζ within certain bounded limits is to be studied. For general purpose computer programs, even "robust" estimation should be given attention.

7) The experimental comparison of this group of techniques is an enormous task and until now, the author only performed serious experiments on anomalous environments (W unspecified), finite set Q-model environments, and deterministic environments.

8) The stochastic analog of creeping random search [7], [11], [19], [29] for the local hill-climbing of continuous $Q(w)$ has been studied by Poznyak [12], but the conditions imposed on the noise are not realistic. It seems possible to derive inequalities in the W domain using our approach from which the convergence can be proved—under the most general conditions—using the theorems of Braverman and Rozonoer [9], [22].

Appendix

In the proofs, some inequalities for sums of independent random variables will be needed. Consider that ζ_j is the average over λ_j iid random variables with pdf $\pi(\zeta)$ and mean μ_1. It is desired to derive a lower bound for Prob $\{|\zeta_j - \mu_1| \leq \varepsilon\}$, $\varepsilon > 0$. Three environments are considered.

1) Q-model environment: If all the moments μ_k up to $k \geq 2$ (k even) exist, then the Markov inequality reads:

$$\text{Prob}\,\{|\zeta_j - \mu_1| \leq \varepsilon\} \geq 1 - \frac{L_k \cdot \mu_k}{\lambda_j^{k/2} \cdot \varepsilon^k} \qquad \text{(A1)}$$

where L_k is a constant only depending upon k. Also, if μ_2 exists:

$$\text{Prob}\,\{|\zeta_j - \mu_1| \leq \varepsilon\} \geq 1 - \frac{L_1 \cdot \mu_1^*}{\lambda_j \cdot \varepsilon} \geq 1 - \frac{\sqrt{\mu_2}}{\lambda_j \cdot \varepsilon} \qquad \text{(A2)}$$

where

$$\mu_1^* \triangleq E\{|\zeta - \mu_1|\} = \int |\zeta - \mu_1| \cdot \pi(\zeta)\, d\zeta$$

in view of $\mu_1^* \leq \sqrt{\mu_2}$ and $L_1 = 1$.

2) *P-model environment* ($\zeta \in \{0,1\}$) and *3) S-model environment* ($\zeta \in [0,1]$): Hoeffding's inequality for sums of iid bounded random variables reads:

$$\text{Prob}\,\{|\zeta_j - \mu_1| > \varepsilon\} \leq 2e^{-2\lambda_j \varepsilon^2} \qquad \text{(A3)}$$

If $\zeta \in [A,B]$, $|A| < \infty$, $|B| < \infty$, (A4) holds:

$$\text{Prob}\{|\zeta_j - \mu_1| > \varepsilon\} \le 2e^{-2\lambda_j(\varepsilon/(B-A))^2} \quad \text{(A4)}$$

Proof of Theorem 1

$Z(X_j)$ defined in (32) is maximal for $p(u/X_j) = \delta(u - \theta)$, where θ is specially selected to maximize $Z(X_j)$. Thus

$$Z_j = \sup_{X_j \in X} Z(X_j) = \sup_{\theta} (q_j - \theta) \cdot E\{\phi(q_j,\theta) - \xi^*(q_j,\theta)\}. \quad \text{(A5)}$$

Consider three cases:

1) $\theta > q_j$
2) $q_j \ge \theta > q_j - \varepsilon_j - \rho$
3) $q_j - \varepsilon_j - \rho \ge \theta$

where ρ is a freely chosen positive quantity (which may depend upon j).

Case 1: $\theta > q_j$: $\phi(q_j,\theta) = 0$.

1a) Deterministic environment: $\mu_2 = 0$, and thus

$$\xi^*(q_j,\theta) = \phi(q_j,\theta) = Z(X_j) = 0.$$

1b) Random environment: There holds

$$\begin{aligned} E\{\xi^*(q_j,\theta)\} &\le \text{Prob}\left\{\zeta_j < q_j - \frac{\varepsilon_j + \theta - q_j}{2}\right\} + \text{Prob}\left\{\zeta_j^* > \theta + \frac{\varepsilon_j + \theta - q_j}{2}\right\} \\ &\le \text{Prob}\left\{|\zeta_j - q_j| \ge \frac{\varepsilon_j + \theta - q_j}{2}\right\} + \text{Prob}\left\{|\zeta_j^* - \theta| \ge \frac{\varepsilon_j + \theta - q_j}{2}\right\} \end{aligned} \quad \text{(A6)}$$

where ζ_j and ζ_j^* are averages over λ_{Bj} and λ_{Tj} iid random variables with pdf $\pi(\zeta/w_j)$ and $\pi(\zeta/w_{j+1}^*)$ and respective means $Q(w_j) = q_j$ and $Q(w_{j+1}^*) = \theta$. By definition of $E\{\xi^*(q_j,\theta)\}$ and the inequalities (A1)–(A4), (A6) can be sharpened: first, for Q-model environments with μ_k existing ($k \ge 2$), (A1) and (A2) hold. Thus

$$Z_j = \sup_{\theta} (\theta - q_j) \cdot \frac{L_k \cdot \mu_k}{\left(\frac{\varepsilon_j + \theta - q_j}{2}\right)^k} \cdot (\lambda_{Bj}^{-k/2} + \lambda_{Tj}^{-k/2}). \quad \text{(A7)}$$

Since μ_2 exists, (A7) also holds for $k = 1$, in which case μ_1 has to be replaced by $\sqrt{\mu_2}$. Expression (A7) is maximal for

$$\theta = \begin{cases} q_j + \dfrac{\varepsilon_j}{k-1}, & \text{for } k = 2,4,6,8,\cdots \\ \infty, & \text{for } k = 1. \end{cases} \quad \text{(A8)}$$

Clearly, substituting θ in (A7) by its value computed in (A8), one obtains

$$Z_j = \frac{L_k \cdot \mu_k}{\left(\frac{k}{2}\right)^k \cdot \left(\frac{\varepsilon_j}{k-1}\right)^{k-1}} \cdot (\lambda_{Bj}^{-k/2} + \lambda_{Tj}^{-k/2}). \quad \text{(A9)}$$

Secondly, for P and S-model environments, using (A3)–(A4):

$$Z_j = \sup_{\theta} (\theta - q_j) \cdot 2(e^{-\lambda_{Bj}\cdot(\varepsilon_j+\theta-q_j)^2/2} + e^{-\lambda_{Tj}\cdot(\varepsilon_j+\theta-q_j)^2/2}). \quad \text{(A10)}$$

Consider the first term in (A10) with $a = \sqrt{\lambda_{Bj}/2}$, $b = a\varepsilon_j$ and $x = \theta - q_j > 0$. Since $xe^{-(ax+b)^2}$ has a maximal value for $ax + b = -1/2ax$ of $xe^{-1/4a^2x^2}$ and since the positive root x_0 of $(ax)^2 + b(ax) - 1 = 0$ is $x_0 = (b/a) \cdot (\sqrt{1 + 2/b^2} - 1) \le (1/ab)$, we have by the monotonicity of $e^{-1/4a^2x^2}$ that the first term is always smaller than $(ab)^{-1}e^{-b^2/4}$. Maximizing the second term separately gives

$$Z_j = 4(\lambda_{Bj}\varepsilon_j)^{-1}e^{-\lambda_{Bj}\varepsilon_j^2/8} + 4(\lambda_{Tj}\varepsilon_j)^{-1}e^{-\lambda_{Tj}\varepsilon_j^2/8}. \quad \text{(A11)}$$

Case 2: $q_j \ge \theta > q_j - \varepsilon_j - \rho$; $\phi(q_j,\theta) = 1$. No sharper bound can be found for $E\{\xi^*(q_j,\theta)\}$ than $E\{\xi^*(q_j,\theta)\} \ge 0$. For all the environments, we have

$$Z_j = \sup_{\theta} (q_j - \theta) = 2\varepsilon_j \quad \text{(A12)}$$

because $q_j - \theta < \varepsilon_j + \rho$ in the given interval and $\rho = \varepsilon_j$.

Case 3: $\theta \le q_j - \varepsilon_j - \rho$, $\rho = \varepsilon_j$ as in Case 2. Furthermore, by definition, $\phi(q_j,\theta) = 1$:

$$E\{\xi^*(q_j,\theta)\} \ge \text{Prob}\left\{|\zeta_j - q_j| \le \frac{q_j - \theta + \varepsilon_j}{2}\right\} \cdot \text{Prob}\left\{|\zeta_j^* - \theta| \le \frac{q_j - \theta + \varepsilon_j}{2}\right\}. \quad \text{(A13)}$$

3a) Deterministic environment: Clearly, $Z(X_j) = 0$.

3b) Q-model random environment: Replacing the probabilities in (A13) by the bounds given in (A1) and (A2), there holds:

$$E\{\xi^*(q_j,\theta)\} \ge \left(1 - \frac{L_k\mu_k}{\left(\frac{q_j - \theta + \varepsilon_j}{2}\right)^k \cdot \lambda_{Tj}^{k/2}}\right) \cdot \left(1 - \frac{L_k\mu_k}{\left(\frac{q_j - \theta + \varepsilon_j}{2}\right)^k \cdot \lambda_{Bj}^{k/2}}\right).$$

Disregarding the product term and substituting the last expression in (A5) yields

$$Z_j = \sup_{\theta} \frac{(q_j - \theta)L_k\mu_k}{\left(\frac{q_j - \theta + \varepsilon_j}{2}\right)^k} \cdot (\lambda_{Tj}^{-k/2} + \lambda_{Bj}^{-k/2}) \quad \text{(A14)}$$

which is maximal on the boundary of the interval, i.e., for $q_j - \theta = 2\varepsilon_j$ since $\rho = \varepsilon_j$. Thus

$$Z_j = \frac{L_k\mu_k}{\varepsilon_j^{k-1}} \cdot \left(\frac{2^{k+1}}{3^k}\right) \cdot (\lambda_{Tj}^{-k/2} + \lambda_{Bj}^{-k/2}) \quad \text{(A15)}$$

3c) S,P-model environments: Using (A3), by a similar procedure as under 3b):

$$Z_j = \sup_{\theta} (q_j - \theta)2(e^{-(q_j-\theta+\varepsilon_j)^2\cdot\lambda_{Bj}/2} + e^{-(q_j-\theta+\varepsilon_j)^2\cdot\lambda_{Tj}/2}) \quad \text{(A16)}$$

and (A16) is never greater than (A17):

$$Z_j = 4\varepsilon_j^{-1} \cdot (\lambda_{Bj}^{-1} \cdot e^{-\varepsilon_j \lambda_{Bj}^2/8} + \lambda_{Tj}^{-1} \cdot e^{-\varepsilon_j \lambda_{Tj}^2/8}). \quad (A17)$$

Together, safe upper limits Z_j are obtained by summing the expressions in each interval. For S,P-model environments, (A17) is identical to (A11) and can be dropped. Thus, for deterministic environments:

$$Z_j = 2\varepsilon_j$$

for Q-model random environments:

$$Z_j = B_k \left(\frac{1}{\varepsilon_j^{k-1} \cdot \lambda_{Bj}^{k/2}} + \frac{1}{\varepsilon_j^{k-1} \cdot \lambda_{Tj}^{k/2}} \right) + 2\varepsilon_j$$

$$B_k = L_k \cdot \mu_k \cdot \max \left\{ (k-1)^{k-1} \cdot \left(\frac{k}{2}\right)^{-k}; \quad 2^{k+1} 3^{-k} \right\} \quad (A18)$$

and for S,P-model random environments:

$$Z_j = 8\varepsilon_j^{-1} \cdot (\lambda_{Bj}^{-1} \cdot e^{-\varepsilon_j \lambda_{Bj}^2/8} + \lambda_{Tj}^{-1} \cdot e^{-\varepsilon_j \lambda_{Tj}^2/8}) + 2\varepsilon_j.$$

Equation (A18) together with (33) leads to the conditions formulated in Theorem 1.

Proof of Theorem 2

Notice that

$$E\left\{\xi^*(q_j,\theta)/q_j > q_{\min} + \eta, \theta \le q_{\min} + \frac{\eta}{2}\right\}$$
$$\ge \text{Prob}\left\{|\zeta_j - q_j| \le \frac{(\eta/2) - \varepsilon_j}{2}\right\} \cdot \text{Prob}\left\{|\zeta_j^* - \theta| \le \frac{(\eta/2) - \varepsilon_j}{2}\right\}$$
$$\ge \text{Prob}\left\{|\zeta_j - q_j| \le \frac{\eta}{8}\right\} \cdot \text{Prob}\left\{|\zeta_j^* - \theta| \le \frac{\eta}{8}\right\} \cdot \omega\left(\varepsilon_j, \frac{\eta}{4}\right) \quad (A19)$$

$$E\{\xi^*(q_j,\theta)/q_j \le q_{\min} + \eta, \theta > q_{\min} + \eta\}$$
$$\le \text{Prob}\left\{|\zeta_j - q_j| > \frac{\varepsilon_j}{2}\right\} + \text{Prob}\left\{|\zeta_j^* - \theta| > \frac{\varepsilon_j}{2}\right\} \quad (A20)$$

since $\theta \to q_{\min} + \eta$ is the worst case for all the environments (see (A1)–(A3)). Expressions (A19) and (A20) are useful in combination with (39), where Z_{1j} and Z_{2j} are defined. Each type of environment is now considered separately.

1) Deterministic environments: Clearly, (A20) and thus Z_{2j} are 0. The right side in (A19) is equal to $\omega(\varepsilon_j,(\eta/4))$. Condition (41) reads:

$$\sum_{j=1}^{\infty} \alpha_j \cdot \omega\left(\varepsilon_j, \frac{\eta}{4}\right) = \infty \quad (A21)$$

which holds in view of (H3) and (H12), for every $\eta > 0$.

2) Q-model random environments: Combining (A1) and (A2) with (A20) gives a bound for Z_{2j}:

$$Z_{2j} = \frac{L_k \cdot \mu_k}{(\varepsilon_j/2)^k} \cdot (\lambda_{Bj}^{-k/2} + \lambda_{Tj}^{-k/2}). \quad (A22)$$

Combining (A1) and (A2) with (A19) gives a bound for Z_{1j}:

$$Z_{1j} = \left(1 - \frac{L_k \cdot \mu_k}{(\eta/8)^k}\right) \cdot (\lambda_{Tj}^{-k/2} + \lambda_{Bj}^{-k/2}) \cdot \omega\left(\varepsilon_j, \frac{\eta}{4}\right)$$
$$\ge \tfrac{1}{3}\omega\left(\varepsilon_j, \frac{\eta}{4}\right) \cdot \omega\left(\lambda_{Tj}^{-1}, \frac{(\eta/8)^2}{(3L_k \cdot \mu_k)^{2/k}}\right) \cdot \omega\left(\lambda_{Bj}^{-1}, \frac{(\eta/8)^2}{(3L_k \cdot \mu_k)^{2/k}}\right) \quad (A23)$$

and this lower bound tends in the limit to 1/3, since $\varepsilon_j \to 0$, $\lambda_{Bj} \to \infty$, and $\lambda_{Tj} \to \infty$. Using (A22) and (A23), there is no problem in deducing the conditions of convergence stated in Theorem 2 from (41) or (42).

3) S,P-model random environments: Combining (A3) and (A4) with (A20) gives a bound for Z_{2j}:

$$Z_{2j} = 2(e^{-\varepsilon_j^2 \cdot \lambda_{Bj}/2} + e^{-\varepsilon_j^2 \cdot \lambda_{Tj}/2}). \quad (A24)$$

Combining (A3) and (A4) with (A19) gives a bound for Z_{1j}:

$$Z_{1j} \ge (1 - 2e^{-\eta^2 \cdot \lambda_{Bj}/16} - 2e^{-\eta^2 \cdot \lambda_{Tj}/16}) \cdot \omega\left(\varepsilon_j, \frac{\eta}{4}\right)$$
$$\ge \tfrac{1}{3}\omega\left(\varepsilon_j, \frac{\eta}{4}\right) \cdot \omega(\lambda_{Tj}^{-1}, (\eta^2/16 \ln 6)) \cdot \omega(\lambda_{Bj}^{-1}, (\eta^2/16 \ln 6)) \quad (A25)$$

and this lower bound tends in the limit to 1/3, since $\varepsilon_j \to 0$, $\lambda_{Bj} \to \infty$, and $\lambda_{Tj} \to \infty$. Using (A24) and (A25), there is again no problem in deducing the conditions of convergence stated in Theorem 2 from (41) or (42).

As is shown in the text, the convergence, for all $\eta > 0$, of $\omega(q_j, q_{\min} + \eta)$ to 1 is linked with the convergence of $\bar{q}_j$ to $q_{\min}$. This completes the proof.

Proof of (48)

Equation (47) and $E\{f(X_j)\} \to f_0$ imply that

$$E\{f_{Aj}\} = L_j^{-1} \cdot \sum_{i=1}^{j} \lambda_i \cdot E\{f(X_i)\} \to f_0$$

where

$$L_j = \sum_{i=1}^{j} \lambda_i$$

by Toeplitz's lemma (see [10, p. 238]).

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