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# LIMIT LAWS OF ERDÖS–RÉNYI–SHEPP TYPE

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Let $S_n = X_1 + \cdots + X_n$ be the $n$th partial sum of an i.i.d. sequence of random variables. We describe the limiting behavior of

$$T_n = \max_{1 \le i \le n}(S_{i+\kappa(i)} - S_i),$$
$$U_n = \max_{0 \le i \le n-k}(S_{i+k} - S_i),$$
$$W_n = \max_{0 \le i \le n-k}\max_{1 \le j \le k}(S_{i+j} - S_i)$$

and

$$V_n = \max_{0 \le i \le n-k}\min_{1 \le j \le k}(k/j)(S_{i+j} - S_i),$$

for $k = \kappa(n) = [c \log n]$, and where $c > 0$ is a given constant. We assume that the random variables $X_i$ are centered and have a finite moment generating function in a right neighborhood of zero, and obtain among other results the full form of the Erdös–Rényi (1970) and Shepp (1964) theorems. Our conditions extend those of Deheuvels, Devroye and Lynch (1986) to cover a larger class of distributions.

**1. Introduction.** Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables, with partial sums $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$. We are concerned with the limiting behavior of

$$T_n = \max_{1 \le i \le n}\{S_{i+\kappa(i)} - S_i\}, \quad U_n = \max_{0 \le i \le n-k}\{S_{i+k} - S_i\},$$

$$W_n = \max_{0 \le i \le n-k}\max_{1 \le j \le k}\{S_{i+j} - S_i\}, \quad V_n = \max_{0 \le i \le n-k}\min_{1 \le j \le k}(k/j)\{S_{i+j} - S_i\},$$

where $k = \kappa(n) = [c \log n]$, with $[u] \le u < [u] + 1$ denoting the integer part of $u$, and where $c > 0$ is fixed.

We shall assume throughout that the following conditions are satisfied:

(A) $E(X_1) = 0$.
(B) $X_1$ is nondegenerate, i.e., $P(X_1 = x) < 1$ for all $x$.
(C) $t_0 = \sup\{t;\ \phi(t) = E(\exp(tX_1)) < \infty\} > 0$.

We will use in the sequel the notation of Deheuvels, Devroye and Lynch (1986), and define the following quantities:

$$m(t) = \phi'(t)/\phi(t), \qquad \sigma^2(t) = m'(t), \qquad 0 \le t < t_0;$$

$$A = \lim_{t \uparrow t_0} m(t), \qquad c_0 = 1\bigg/\int_0^{t_0} tm'(t)\,dt;$$

$$\rho = \rho(\alpha) = \exp(-1/c) = \inf_t \phi(t)e^{-t\alpha} = \phi(t^\star)e^{-t^\star\alpha}, \qquad m(t^\star) = \alpha,$$

$$0 < t^\star = t^\star(\alpha) < t_0, \quad 0 < \alpha < A, \qquad 0 \le c_0 < c = c(\alpha) < \infty.$$

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Note that $c = c(\alpha)$ [resp. $t^\star = t^\star(\alpha)$] is a decreasing (resp. increasing) continuous one-to-one function of $\alpha \in (0, A)$, such that

$$\lim_{\alpha \downarrow 0} c(\alpha) = \infty, \qquad \lim_{\alpha \uparrow A} c(\alpha) = c_0, \qquad \lim_{\alpha \downarrow 0} t^\star(\alpha) = 0, \qquad \lim_{\alpha \uparrow A} t^\star(\alpha) = t_0.$$

Limiting results concerning increments of partial sums such as $U_n$ with the "critical" choice $k = [c \log n]$ are usually called Erdös–Rényi laws after the best-known theorem of this type due to Erdös and Rényi (1970) who proved that, for any $c_0 < c < \infty$,

$$\lim_{n \to \infty} (\alpha k)^{-1} U_n = 1 \quad \text{almost surely.}$$

Earlier, Shepp (1964) had proved, under the same assumptions, that

$$\lim_{n \to \infty} (\alpha k)^{-1} T_n = 1 \quad \text{almost surely.}$$

These results have been refined by S. Csörgő (1979) and by M. Csörgő and Steinebach (1981) who showed that, for any $c_0 < c < \infty$,

$$(\alpha k)^{-1} U_n = 1 + o(k^{-1/2}) \quad \text{almost surely}$$

and

$$(\alpha k)^{-1} W_n = 1 + o(k^{-1/2}) \quad \text{almost surely.}$$

The exact rate of convergence of $(\alpha k)^{-1}U_n$ and $(\alpha k)^{-1}T_n$ has been obtained by Deheuvels, Devroye and Lynch (1986). This is given in Theorem A.

THEOREM A. *Let $c_0 < c < \infty$. Then we have*

(i) $(U_n - \alpha k)/\log k \to -(2t^\star)^{-1}$ *in probability;*
(ii) $\limsup_{n \to \infty} (U_n - \alpha k)/\log k = (2t^\star)^{-1}$ *almost surely;*
(iii) $\liminf_{n \to \infty} (U_n - \alpha k)/\log k = -(2t^\star)^{-1}$ *almost surely.*

*In statements* (i)–(iii), $U_n$ *can be replaced by* $T_n$.

The aim of this paper is threefold. In the first place, we consider an arbitrary $c \in (c_0, \infty)$, and we obtain versions of Theorem A valid for $W_n$ and $V_n$. This is achieved in Sections 3 and 4, while in Section 2, we give a large deviation estimate which is used in the proofs.

Secondly, we consider the case where $0 < c \le c_0$. This corresponds to the so-called full form of the Erdös–Rényi theorem, which covers specific distributions characterized in Theorem B, due to Deheuvels, Devroye and Lynch (1986):

THEOREM B. $c_0 = 0$ *in all cases except the following two:*

(i) $A < \infty$, $t_0 < \infty$. *In this case,* $\operatorname{ess\,sup} X_1 = \infty$ *and*

$$c_0 = 1/(A t_0 - \log \phi(t_0)).$$

(ii) $A < \infty$, $t_0 = \infty$. *In this case,* $\operatorname{ess\,sup} X_1 = A$, $P(X = A) > 0$ *and*

$$c_0 = -1/\log P(X_1 = A).$$

We obtain in Section 5 the limiting behavior of $T_n$, $U_n$, $V_n$ and $W_n$, corresponding to cases (i)–(ii) of Theorem B.

Thirdly, we investigate the limiting distribution of $U_n$, which is evaluated in Section 6.

In Section 7, we discuss some applications, with emphasis on Brownian motion and empirical processes.

Before proceeding with the details of our theorems, we remark that, as will be shown in the sequel, in all cases, the random variables $T_n$, $U_n$, $V_n$ and $W_n$ remain asymptotically very close to each other. This result is somewhat surprising for $V_n$ which one could have expected to be much smaller that $U_n$ and $W_n$.

General references on Erdös–Rényi–Shepp type theorems are to be found in S. Csörgő (1979) and Deheuvels (1985). It is worthwhile mentioning that if $l = l_n = [\tilde{l}_n]$, where $\{\tilde{l}_n, n \geq 1\}$ is a real-valued nondecreasing sequence such that

(i) $1 \leq \tilde{l}_n \leq n$, $n = 1, 2, \dots;$
(ii) $\tilde{l}_n/n \downarrow$; $\tilde{l}_n/\log n \to \infty$;
(iii) $\{\log(n/\tilde{l}_n)\}/\log\log n \to \infty$;

assuming, in addition to (A)–(C) that $\phi(t) < \infty$ in a neighborhood of zero, and setting $\sigma^2 = \sigma^2(0) = E(X_1^2) < \infty$, we have

$$\lim_{n \to \infty} \{2l\sigma^2 \log(n/l)\}^{-1/2} \max_{0 \leq i \leq n-l} \{S_{i+l} - S_i\} = 1 \quad \text{almost surely.}$$

This shows the interest of Erdös–Rényi–Shepp type increments for which the normalizing factor $(\alpha k)^{-1}$ characterizes the distribution of $X_1$, while $(2l\sigma^2\log(n/l))^{-1/2}$ depends upon this distribution through $\sigma^2 = E(X_1^2)$ only.

Finally, we note that our methods can be extended to cover the case where $k = \kappa(n) = c \log n + o(\log\log n)$, as $n \to \infty$. Expansions for $k/\log n \to \infty$ are to be found in Deheuvels and Steinebach (1986).

## 2. Large deviations estimates.

We shall make use of the following theorem due to Petrov (1965).

THEOREM C (Petrov, 1965).

$$P(S_n \geq n\alpha) \sim \frac{\psi(t^\star)}{\sqrt{n}}\rho^n = \frac{\psi(t^\star)}{\sqrt{n}}\exp\left(-\frac{n}{c}\right) = \frac{\psi(t^\star)}{\sqrt{n}}\exp(n(\log\phi(t^\star) - t^\star\alpha)),$$

*uniformly for* $\alpha \in [\varepsilon, \min(A - \varepsilon, 1/\varepsilon)]$, *where* $\varepsilon > 0$ *is arbitrary, and* $\psi(t^\star) > 0$ *is a finite number depending upon* $t^\star$ *and the distribution of* $X_1$ *only. For nonlattice distributions, one can take* $\psi(t^\star) = (t^\star\sigma(t^\star)\sqrt{2\pi})^{-1}$, *while for lattice distributions with span H, one can take* $\psi(t^\star) = H(\sigma(t^\star)\sqrt{2\pi}(1 - e^{-Ht^\star}))^{-1}$.

We shall prove the following large deviation result which has interest in itself.

THEOREM 1.

$$P(S_1 \geq \alpha, S_2 \geq 2\alpha, \dots, S_n \geq n\alpha) \sim \frac{\psi(t^\star)}{n\sqrt{n}}\rho^n \exp\left\{\sum_{k=1}^{+\infty} \frac{P(S_k \geq k\alpha)}{k\rho^k}\right\}, \quad \textit{as } n \to \infty,$$

*uniformly for* $\alpha \in [\varepsilon, \min(A - \varepsilon, 1/\varepsilon)]$, *where* $\varepsilon > 0$ *is arbitrary, and where* $\psi(t^\star)$ *is defined as in Theorem* C.

We shall make use of the following lemma, due to Sparre-Andersen (1953/1954) [see, e.g., Stout (1974), page 342].

LEMMA A. *Let* $\alpha_0 = \beta_0 = 1$ *and for* $n \geq 1$, $\alpha_n = P(S_1 \geq \alpha, S_2 \geq 2\alpha, \dots, S_n \geq n\alpha)$ *and* $\beta_n = P(S_n \geq n\alpha)$. *Then, for any* $|s| < 1$, *we have*

$$\sum_{n=0}^{+\infty} \alpha_n s^n = \exp\left( \sum_{n=1}^{+\infty} \beta_n \frac{s^n}{n} \right).$$

PROOF OF THEOREM 1. By Lemma A, we have the straightforward expansion

$$\alpha_n = \frac{\beta_n}{n} + \frac{1}{2} \sum_{\substack{n_1, n_2 \geq 1 \\ n_1 + n_2 = n}} \frac{\beta_{n_1}\beta_{n_2}}{n_1 n_2} + \cdots + \frac{1}{k!} \sum_{\substack{n_1, \dots, n_k \geq 1 \\ n_1 + \cdots + n_k = n}} \frac{\beta_{n_1} \cdots \beta_{n_k}}{n_1 \cdots n_k} + \cdots .$$

For $1 \leq i \leq k \leq n$, let

$$A_i(k, n) = \{(n_1, \dots, n_k): n_1 + \cdots + n_k = n,\ n_1, \dots, n_k \geq 1,\\ n_i > n_j,\ j < i,\ n_i \geq n_l,\ l > i\}.$$

Let $B_i(k, n)$ be the image of $A_i(k, n)$ under the map $(n_1, \dots, n_k) \to (n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_k)$. We have evidently

$$\begin{aligned} n^{3/2}\rho^{-n}\alpha_n &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{k} \sum_{A_i(k,n)} \frac{\beta_{n_1} \cdots \beta_{n_k}}{n_1 \cdots n_k} n^{3/2}\rho^{-n} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{k} \sum_{B_i(k,n)} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{\beta_{n_j}}{n_j \rho^{n_j}} \right\} \{n_i/n\}^{-3/2} \Phi(n_i), \end{aligned}$$

where $\Phi(m) = \beta_m \sqrt{m}\, \rho^{-m}$.

By Theorem C, we know that, for all $m$ and $\alpha \in [\varepsilon, \min(A - \varepsilon, 1/\varepsilon)]$, $\Phi(m) \leq \Phi < \infty$, where $\Phi$ is an appropriate constant. Furthermore, on $B_i(k, n)$,

$$n_i/n \geq 1/k.$$

Therefore, the above sum is dominated by the series

$$\Phi \sum_{k=1}^{\infty} \frac{1}{k!} k^{3/2} \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} \left\{ \sum_{n_j \geq 1} \frac{\beta_{n_j}}{n_j \rho^{n_j}} \right\} = \Phi \sum_{k=1}^{\infty} \frac{1}{k!} k^{5/2} \left( \sum_{n=1}^{\infty} \frac{\beta_n}{n\rho^n} \right)^{k-1} < \infty.$$

Using again Theorem C, we see that, uniformly in $\alpha \in [\varepsilon, \min(1 - \varepsilon, 1/\varepsilon)]$,

$$\sum_{n=1}^{\infty} \frac{\beta_n}{n\rho^n} \leq \Theta < \infty, \quad \text{for some constant } \Theta.$$

It follows that the above series is in turn dominated by

$$\Phi \sum_{k=1}^{\infty} \frac{1}{k!} k^{5/2} \Theta^{k-1} < \infty.$$

Passing to the limit term-by-term, we see that, uniformly in $\alpha \in [\varepsilon, \min(A - \varepsilon, 1/\varepsilon)]$,

$$\lim_{n\to\infty} n^{3/2} \rho^{-n} \alpha_n = \psi(t^\star) \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{k} \left( \sum_{n=1}^{\infty} \frac{\beta_n}{n\rho^n} \right)^{k-1} = \psi(t^\star) \exp\left( \sum_{n=1}^{\infty} \frac{\beta_n}{n\rho^n} \right),$$

as sought. Here, we have used again Theorem C and the dominated convergence theorem. This proves Theorem 1. □

Proceeding as in Deheuvels, Devroye and Lynch (1986), Corollaries 1 and 2, we obtain the following Corollaries of Theorem 1.

COROLLARY 1. *Let $\alpha \in (0, A)$ and let $y_n$ be a sequence of numbers satisfying $ny_n^2 \to 0$ as $n \to \infty$. Then, uniformly over all sequences $z_n$ with $|z_n| \le |y_n|$, we have*

$$P(S_n \ge n(\alpha + z_n)) \sim \frac{\psi(t^\star)}{\sqrt{n}} \exp\left(-\frac{n}{c}\right) \exp(-nz_n t^\star)$$

*and*

$$P(S_1 \ge \alpha + z_n, S_2 \ge 2(\alpha + z_n), \dots, S_n \ge n(\alpha + z_n))$$
$$\sim \frac{\psi(t^\star)}{n\sqrt{n}} \exp\left(-\frac{n}{c}\right) \exp(-nz_n t^\star) \Delta(\alpha + z_n),$$

*where $\Delta(\lambda) = \exp\{\sum_{k=1}^{\infty}(P(S_k \ge k\lambda))/(k\rho^k(\lambda))\}$. Furthermore, we have*

$$0 < \exp\left\{ \sum_{k=1}^{\infty} \frac{P(S_k > k\alpha)}{k\rho^k} \right\} \le \liminf_{n\to\infty} \Delta(\alpha + z_n) \le \limsup_{n\to\infty} \Delta(\alpha + z_n)$$
$$\le \exp\left\{ \sum_{k=1}^{\infty} \frac{P(S_k \ge k\alpha)}{k\rho^k} \right\} < \infty.$$

PROOF. Corollary 1 of Deheuvels, Devroye and Lynch (1986) gives the first statement, while the second follows from the first by Theorem 1 and the observation that, as $n \to \infty$, $P(S_1 \ge \lambda, S_2 \ge 2\lambda, \dots, S_n \ge n\lambda)(\Delta(\lambda))^{-1} \sim n^{-1}P(S_n \ge n\lambda)$ uniformly for $\lambda = \alpha + z_n$. Finally, note that $z_n \to 0$, $\rho(\alpha + z_n) \to \rho(\alpha)$ as $n \to \infty$, and, for any fixed $k \ge 1$,

$$P(S_k > k\alpha) \le \liminf_{n\to\infty} P(S_k \ge k(\alpha + z_n))$$
$$\le \limsup_{n\to\infty} P(S_k \ge k(\alpha + z_n)) \le P(S_k \ge k\alpha).$$

The conclusion follows by the dominated convergence theorem. □

COROLLARY 2. *For all* $\varepsilon \in \mathbb{R}$ *and* $\alpha \in (0, A)$, *we have*

$$(\psi(t^\star) + o(1))k^{-(1/2)-(\pm(1/2)+\varepsilon)} \le nP\left(S_k \ge \alpha k + \left(\pm\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{k}\right)$$
$$\le \left(e^{1/c}\psi(t^\star) + o(1)\right)k^{-(1/2)-(\pm(1/2)+\varepsilon)},$$

*and there exist constants* $0 < C_1 = C_1(\alpha) \le C_2 = C_2(\alpha) < \infty$, *such that*

$$(C_1 + o(1))k^{-(1/2)-(\pm(1/2)+\varepsilon)} \le nP\left(\min_{1\le j\le k}\frac{kS_j}{j} \ge \alpha k + \left(-1 \pm \tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star}\right)$$
$$\le (C_2 + o(1))k^{-(1/2)-(\pm(1/2)+\varepsilon)}.$$

PROOF. It follows directly from Corollary 1 and the observation that $e^{(k+1)/c} \ge n \ge e^{k/c}$. $\square$

REMARK 1. Lemma A yields the simple inequalities

$$\frac{P(S_n \ge n\alpha)}{n} \le P(S_1 \ge \alpha, S_2 \ge 2\alpha, \dots, S_n \ge n\alpha) \le P(S_n \ge n\alpha),$$

which suffice, jointly with Theorem C, for a Chernoff-type large deviation result of the form

$$\lim_{n\to\infty} n^{-1}\log P(S_1 \ge \alpha, S_2 \ge 2\alpha, \dots, S_n \ge n\alpha) = \lim_{n\to\infty} n^{-1}\log P(S_n \ge n\alpha)$$
$$= \log \rho.$$

REMARK 2. In Theorems C and 1 (resp. Corollaries 1 and 2), it is implicitly assumed in case where $X_1$ follows a lattice distribution, that $n\alpha$ [resp. $n(\alpha + z_n)$] belongs to the subgroup of $\mathbb{R}$ generated by the support of the distribution of $X_1$. Such a simplification does not affect the proofs of our theorems in the sequel, with the only exception of Section 6, where the assumption that the distribution is nonlattice is essential in the proofs.

**3. The random variable $W_n$.** Throughout, we shall make use of the sequence of integers defined by

$$n_j = \inf\{n; [c\log n] = j\}.$$

We note in the first place that $V_n \le U_n \le W_n$. In this section, we assume that $\alpha \in (0, A)$. A straightforward application of Theorem A proves our first lemma stated below.

LEMMA 1.

(i) $\limsup_{n\to\infty}(W_n - \alpha k)/\log k \ge (2t^\star)^{-1}$ *almost surely.*
(ii) $\liminf_{n\to\infty}(W_n - \alpha k)/\log k \ge -(2t^\star)^{-1}$ *almost surely.*
(iii) *For any* $\varepsilon > 0$, *we have, as* $n \to \infty$,

$$P\left((W_n - \alpha k)/\log k \ge -(2t^\star)^{-1} - \varepsilon\right) \to 1.$$

LEMMA 2. *For all $\varepsilon > 0$, we have, as $n \to \infty$,*

$$P\left(W_n \geq \alpha k + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star}\right) \leq \frac{e^{1/c}\psi(t^\star) + o(1)}{k^\varepsilon}\frac{\phi(t^\star)}{\phi(t^\star) - 1}.$$

PROOF. By Bonferroni, we have

$$P\left(W_n \geq \alpha k + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star}\right) \leq n\sum_{j=1}^{k} P\left(S_j \geq \alpha k + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star}\right)$$
$$= P_1 + P_2,$$

where

$$P_1 = n\sum_{j=1}^{k-m}(\cdot) \quad \text{and} \quad P_2 = n\sum_{j=k-m+1}^{k}(\cdot), \qquad m = [k^{1/3}].$$

Noting that $m^2 = o(k)$, we have by Corollaries 1 and 2, uniformly in $k - m + 1 \leq j \leq k$,

$$nP\left(S_j \geq \alpha k + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star}\right) = nP\left(S_j \geq \alpha j + \alpha(k-j) + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star}\right)$$
$$\sim n\frac{\psi(t^\star)}{\sqrt{j}}e^{-j/c}\exp(-\alpha(k-j)t^\star)k^{(1/2)-\varepsilon}$$
$$\sim n\psi(t^\star)e^{-k/c}\exp\left(-(k-j)\left(\alpha t^\star - \frac{1}{c}\right)\right)k^{-\varepsilon}$$
$$\leq e^{1/c}k^{-\varepsilon}\psi(t^\star)\phi(t^\star)^{-(k-j)},$$

which, if summed over $j = k - m + 1$ to $j = k$ gives a number not exceeding

$$\left(e^{1/c}\psi(t^\star) + o(1)\right)k^{-\varepsilon}(1 - 1/\phi(t^\star))^{-1}.$$

By Jensen's inequality we also have, for $1 \leq j \leq k - m$,

$$nP\left(S_j \geq \alpha k + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star}\right) \leq n\phi(t^\star)^j e^{-t^\star \alpha k}k^{(1/2)-\varepsilon}$$
$$\leq e^{(k+1)/c}\phi(t^\star)^{k-m}e^{-t^\star\alpha k}k^{1/2}$$
$$= e^{1/c}\phi(t^\star)^{-m}\sqrt{k}.$$

Summed over $j = 1$ to $j = k - m$, the upper bound is

$$O\left(k^{3/2}\exp(-[k^{1/3}]\log\phi(t^\star))\right) = o(k^{-\varepsilon})$$

and we are done. □

LEMMA 3. *For any $\varepsilon > 0$, we have, as $n \to \infty$,*

$$P\left((W_n - \alpha k)/\log k \leq -(2t^\star)^{-1} + \varepsilon\right) \to 1.$$

PROOF. It follows directly from Lemma 2. □

The main theorems of this section follow.

**Theorem 2.** *For any $\alpha \in (0, A)$ or equivalently, for any $c = c(\alpha) \in (c_0, \infty)$, we have, as $n \to \infty$,*

$$(W_n - \alpha k)/\log k \to -(2t^\star)^{-1}, \quad \textit{in probability}.$$

**Proof.** Combine Lemmas 1 and 3. □

**Theorem 3.** *For any $\alpha \in (0, A)$ or equivalently, for any $c = c(\alpha) \in (c_0, \infty)$, we have*

$$\text{(i)} \qquad \limsup_{n\to\infty} (W_n - \alpha k)/\log k = (2t^\star)^{-1} \quad \textit{almost surely};$$

$$\text{(ii)} \qquad \liminf_{n\to\infty} (W_n - \alpha k)/\log k = -(2t^\star)^{-1} \quad \textit{almost surely}.$$

**Proof.** (ii) follows from Lemma 1(ii) and Theorem 2, noting that if $P(A_n) \to 1$, then $P(A_n \text{ i.o.}) = 1$.

For the proof of (i), we remark that $W_n$ is nondecreasing in $n$ and that $W_n \le W_{n_{j+1}-1}$ for $n_j \le n < n_{j+1}$. Thus, for $\varepsilon > 0$,

$$P\left(W_n \ge \alpha k + \left(\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star} \text{ i.o.}\right) \le P\left(W_{n_{j+1}-1} \ge \alpha j + \left(\tfrac{1}{2} + \varepsilon\right)\frac{\log j}{t^\star} \text{ i.o. (in } j)\right),$$

and the latter probability is 0 if the individual probabilities are summable in $j$, by Borel–Cantelli. Now by Lemma 2, with $\varepsilon = \varepsilon' + 1$,

$$P\left(W_{n_{j+1}-1} \ge \alpha j + \left(\tfrac{1}{2} + \varepsilon'\right)\frac{\log j}{t}\right) = O\left(j^{-1-\varepsilon'}\right),$$

which is summable in $j$ for all $\varepsilon' > 0$. In view of Lemma 1(i), this proves Theorem 3. □

**Remark 3.** The limiting behavior in probability and almost surely of $(T_n - \alpha k)/\log k$, $(U_n - \alpha k)/\log k$ and $(W_n - \alpha k)/\log k$ are the same up to the first-order terms.

**4. The random variable $V_n$.** In this section, we shall consider the events denoted by

$$A_i = \{S_{i+1} - S_i \ge \alpha + z_k, S_{i+2} - S_i \ge 2(\alpha + z_k), \dots, S_{i+k} - S_i \ge k(\alpha + z_k)\}, \quad i = 0, 1, \dots,$$

where $z_j$ is a sequence which will be made precise later on.

We assume throughout that $\alpha \in (0, A)$.

**Lemma 4.** *We have*

$$\limsup_{n\to\infty} (V_n - \alpha k)/\log k \le -(2t^\star)^{-1} \quad \textit{almost surely}.$$

PROOF. We note that $V_n \le V_{n_{j+1}-1}$ for $n_j \le n < n_{j+1}$. Thus, for $\varepsilon > 0$,

$$P\left(V_n \ge \alpha k + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log k}{t^\star} \text{ i.o.}\right)$$
$$\le P\left(V_{n_{j+1}-1} \ge \alpha j + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log j}{t^\star} \text{ i.o. (in } j)\right).$$

By Corollary 2 we see, using the same argument as in Theorem 3, that, for any $\varepsilon > 0$,

$$P\left(V_{n_{j+1}-1} \ge \alpha j + \left(-\tfrac{1}{2} + \varepsilon\right)\frac{\log j}{t^\star}\right) = o\left(j^{-1-\varepsilon}\right),$$

which is summable in $j$. The result follows by Borel–Cantelli. □

LEMMA 5. *For any $\varepsilon > 0$, we have, as $n \to \infty$,*

$$P\left((V_n - \alpha k)/\log k \le -3(2t^\star)^{-1} + \varepsilon\right) \to 1.$$

PROOF. By Bonferroni, we have

$$P\left((V_n - \alpha k)/\log k \ge -3(2t^\star)^{-1} + \varepsilon\right) = P\left(\bigcup_{i=0}^{n-k} A_i\right) \le nP(A_0),$$

with $z_k = (-\tfrac{3}{2} + \varepsilon')(\log k)/(kt^\star)$, $\varepsilon = \varepsilon'/t^\star$. By Corollary 2 this probability tends to 0, hence the result follows. □

LEMMA 6. *We have*

$$\liminf_{n\to\infty} (V_n - \alpha k)/\log k \le -3(2t^\star)^{-1} \quad \textit{almost surely}.$$

PROOF. It follows from Lemma 5, by the same arguments as in Theorem 3. □

Let us now introduce the following notation: For any $\lambda > 0$, put

$$\tau_0(\lambda) = 0, \qquad \tau_1(\lambda) = \min\{n > 0;\ S_n < n\lambda\}.$$

Since $n^{-1}S_n \to 0$ as $n \to \infty$, the set $\{n > 0;\ S_n < n\lambda\}$ is nonempty and $\tau_1(\lambda)$ is defined a.s. By induction, assuming that $\tau_0(\lambda), \dots, \tau_{j-1}(\lambda)$ have been defined, put

$$\tau_j(\lambda) = \min\left\{n > \tau_{j-1}(\lambda);\ S_n - S_{\tau_{j-1}(\lambda)} < \left(n - \tau_{j-1}(\lambda)\right)\lambda\right\}.$$

It is straightforward that $\tau_0(\lambda) < \tau_1(\lambda) < \cdots$ defines almost surely an increasing sequence of random variables. Let further

$$\theta_j = \theta_j(\lambda) = \tau_j(\lambda) - \tau_{j-1}(\lambda), \qquad j = 1, 2, \dots.$$

The random variables $\theta_1, \theta_2, \dots$ are independent and identically distributed. Finally, put

$$K_\lambda(n) = \max\left\{j \ge 0;\ \tau_j(\lambda) \le n\right\}, \qquad n = 0, 1, \dots.$$

It is noteworthy that, if $i = \tau_{j-1}(\lambda)$, then, for all $l = 0, 1, \dots, \theta_j(\lambda) - 1$, we have

$$S_{i+l} - S_i \geq l\lambda.$$

It follows that, for all $j \geq 1$,

$$P(\theta_j(\lambda) > n) = P(S_1 \geq \lambda, S_2 \geq 2\lambda, \dots, S_n \geq n\lambda), \qquad n = 1, 2, \dots. \qquad \square$$

LEMMA 7. *For any $\lambda > 0$, we have*

$$\Big\{ \max_{1 \leq j \leq K_\lambda(n-k)} \theta_j(\lambda) > k \Big\} \subset \{V_n \geq k\lambda\} \subset \Big\{ \max_{1 \leq j \leq K_\lambda(n-k)+1} \theta_j(\lambda) > k \Big\}.$$

PROOF. We have $\{V_n \geq k\lambda\} = \bigcup_{i=0}^{n-k} A_i$ for $\alpha + z_k = \lambda$. Assume that, for some $i$: $0 \leq i \leq n - k$, the event $A_i$ holds, i.e., we have $S_{i+1} - S_i \geq \lambda, S_{i+2} - S_i \geq 2\lambda, \dots, S_{i+k} - S_i \geq k\lambda$.

There always exists a $j \geq 0$ such that $\tau_j(\lambda) \leq i < \tau_{j+1}(\lambda)$. We must have then $S_i - S_{\tau_j(\lambda)} \geq (i - \tau_j(\lambda))\lambda$, and hence $S_{i+l} - S_{\tau_j(\lambda)} = S_{i+l} - S_i + S_i - S_{\tau_j(\lambda)} \geq (i + l - \tau_j(\lambda))\lambda$ for $l = 1, 2, \dots, k$. This implies that $\tau_{j+1}(\lambda) > i + k$, and hence that $\theta_{j+1}(\lambda) > k$. Note here that $j \leq K_\lambda(n - k)$ which suffices for proof of half of our lemma.

Conversely, if for some $j$: $1 \leq j \leq K_\lambda(n - k)$ we have $\theta_j(\lambda) > k$, then $A_i$ holds for $i = \tau_{j-1}(\lambda)$. The proof is completed by the observation that $i + k < \tau_j(\lambda) \leq n$. Finally, the event $\{\max_{1 \leq j \leq K_\lambda(n-k)} \theta_j(\lambda) > k\}$ is void for $K_\lambda(n - k) = 0$. $\square$

LEMMA 8. *Let $0 < \gamma < \delta < A$. Then, for any $n = 1, 2, \dots$, we have $K_\gamma(n) \leq K_\delta(n)$. Furthermore, there exists an increasing function $g(\gamma)$ of $\gamma \in (0, A)$ such that*

$$\lim_{n\to\infty} n^{-1} K_\gamma(n) = 1/E(\theta_1(\gamma)) = g(\gamma) \quad \textit{almost surely}.$$

PROOF. It is straightforward that $\tau_1(\delta) \leq \tau_1(\gamma)$ and that $\tau_0(\gamma) = \tau_0(\delta) = 0$. Likewise, if $\tau_i(\gamma) \leq \tau_j(\delta) < \tau_{i+1}(\gamma)$, then we must have $\tau_{j+1}(\delta) \leq \tau_{i+1}(\gamma)$. It follows that there exists an $l$ such that $\tau_l(\delta) = \tau_{i+1}(\gamma)$. This implies that the sequences are embedded, i.e., $\{\tau_i(\gamma);\ i \geq 0\} \subset \{\tau_j(\delta);\ j \geq 0\}$, and hence that $K_\gamma(n) \leq K_\delta(n)$. Next, we use the fact that $\{\theta_j(\gamma),\ j \geq 1\}$ is i.i.d. and that

$$E(\theta_1(\gamma)) = \sum_{n=0}^{\infty} P(\theta_1(\gamma) > n) = 1 + \sum_{n=1}^{\infty} P(S_1 \geq \gamma, S_2 \geq 2\gamma, \dots, S_n \geq n\gamma) < \infty,$$

since, for $0 < \gamma < A$, $P(S_1 \geq \gamma, S_2 \geq 2\gamma, \dots, S_n \geq n\gamma) = O(n^{-3/2}\rho^n(\gamma))$ as $n \to \infty$. [Observe that $E(\theta_1(\gamma))\downarrow$ as $\gamma\uparrow$.] This with the elementary renewal theorem completes the proof of Lemma 8. $\square$

LEMMA 9. *Let $z_n \to 0$ as $n \to \infty$. Then, for any $\alpha \in (0, A)$, there exist constants $0 < C < D < \infty$, such that*

$$P\Big( \max_{1 \leq j \leq Cn} \theta_j(\alpha + z_n) > k \quad i.o.\Big) = 1 \Rightarrow P(V_n \geq k(\alpha + z_n) \quad i.o.) = 1$$

*and*

$$P\Big(\max_{1\le j\le Dn}\theta_j(\alpha+z_n)\le k\quad i.o.\Big)=0\Rightarrow P\big(V_n<k(\alpha+z_n)\quad i.o.\big)=0.$$

PROOF. Let $\lambda=\alpha+z_n$. By Lemma 7, $P(\max_{1\le j\le K_\lambda(n-k)}\theta_j(\lambda)>k \text{ i.o.})=1\Rightarrow P(V_n\ge k\lambda \text{ i.o.})=1$. Hence, the first part of Lemma 9 will be proved if we find $0<C<\infty$ such that $P(K_\lambda(n-k)<Cn$ finitely often$)=1$. Take, without loss of generality, $\gamma=\alpha/2<\alpha+z_n$ in Lemma 9. We have then $K_\gamma(n)\le K_\lambda(n)$ and $n^{-1}K_\gamma(n)\to g(\gamma)$ a.s. as $n\to\infty$. If we choose $C=g(\gamma)/2$, we are done. The second part of Lemma 9 can be proved by the same argument, taking now $D=2G(\delta)$, where $\delta$: $\alpha<\delta<A$ is fixed. We have used here the fact that $n-k\sim n$, as $n\to\infty$. □

LEMMA 10. *We have, for any* $\varepsilon>0$,

(i) $$\limsup_{n\to\infty}(V_n-\alpha k)/\log k\ge-(2t^\star)^{-1}\quad \textit{almost surely};$$

(ii) $$\liminf_{n\to\infty}(V_n-\alpha k)/\log k\ge-3(2t^\star)^{-1}\quad \textit{almost surely};$$

(iii) $$P\Big((V_n-\alpha k)/\log k\ge-3(2t^\star)^{-1}-\varepsilon\Big)\to1,\quad \textit{as } n\to\infty.$$

PROOF. Let $\lambda=\alpha+z_n=\alpha+(kt^\star)^{-1}(-\frac{3}{2}-\varepsilon)\log k$. By Corollaries 1 and 2, we have, as $n\to\infty$,

$$P(\theta_1(\lambda)>k)\ge P(S_1\ge\lambda,S_2\ge2\lambda,\dots,S_k\ge k\lambda)\ge(C_1+o(1))k^\varepsilon n^{-1}.$$

It follows that, for a fixed $D>0$,

$$P\Big(\max_{1\le i\le Dn}\theta_i(\lambda)\le k\Big)=\big(1-P(\theta_1(\lambda)>k)\big)^{[Dn]}\le\exp(-[Dn]P(\theta_1(\lambda)>k))$$
$$\le\exp(-DC_1k^\varepsilon(1+o(1))).$$

It follows that, for any $\varepsilon>0$ and $D>0$,

$$\sum_j P\Big(\max_{1\le i\le Dn_j}\theta_i(\alpha+z_{n_j})<j\Big)<\infty.$$

By Borel–Cantelli, this implies that $P(\max_{1\le i\le Dn}\theta_i(\alpha+z_n)\le k \text{ i.o.})=0$. We have used here the fact that $z_n$ is constant for $n_j\le n<n_{j+1}$. By Lemma 9, this suffices for proof of (ii), which implies in turn (iii).

For (i), let $C>0$ be as in Lemma 9, and take

$$\lambda=\alpha+z_n=\alpha+(kt^\star)^{-1}\big(-\tfrac{1}{2}-\varepsilon\big)\log k.$$

By Corollaries 1 and 2, we have now, as $n\to\infty$,

$$P(\theta_1(\lambda)>k)\ge(C_1+o(1))k^{1-\varepsilon}n^{-1}.$$

Let $m_j=[j\max(1,1/C)]$. It follows that

$$\sum_j P\Big(\theta_{[Cm_j]}(\alpha+z_{m_j})>\kappa(m_j)\Big)=\infty,$$

which implies by Borel–Cantelli that $P(\max_{1\le j\le Cn}\theta_j(\alpha+z_n)>k \text{ i.o.})=1$. The proof follows by Lemma 9. □

Combining Lemmas 4, 5, 6 and 10 we have

THEOREM 4. *For any $\alpha\in(0,A)$, or equivalently, for any $c=c(\alpha)\in(c_0,\infty)$, we have*

(i) $$\lim_{n\to\infty}(V_n-\alpha k)/\log k=-3(2t^\star)^{-1}, \quad \textit{in probability};$$

(ii) $$\limsup_{n\to\infty}(V_n-\alpha k)/\log k=-(2t^\star)^{-1} \quad \textit{almost surely};$$

(iii) $$\liminf_{n\to\infty}(V_n-\alpha k)/\log k=-3(2t^\star)^{-1} \quad \textit{almost surely}.$$

REMARK 4. Even though the limiting behavior of $(V_n-\alpha k)/\log k$ differs from that of $(U_n-\alpha k)/\log k$, it is remarkable that

$$\lim_{n\to\infty}k^{-1}T_n=\lim_{n\to\infty}k^{-1}U_n=\lim_{n\to\infty}k^{-1}V_n=\lim_{n\to\infty}k^{-1}W_n=\alpha \quad \text{almost surely}.$$

**5. The full form of the Erdös–Rényi theorem.** In this section, we shall prove the following theorem.

THEOREM 5. *Assume that $c_0>0$. Then, for any $c\in(0,c_0]$, we have*

$$\lim_{n\to\infty}k^{-1}T_n=\lim_{n\to\infty}k^{-1}U_n=\lim_{n\to\infty}k^{-1}V_n=\lim_{n\to\infty}k^{-1}W_n$$
$$=A+\frac{1}{t_0}\left(\frac{1}{c}-\frac{1}{c_0}\right) \quad \textit{almost surely}.$$

PROOF. We shall consider successively case (ii) and case (i) of Theorem B.

CASE (ii). We have $t_0=\infty$, $\operatorname{ess\,sup}X_1=A$, $P(X=A)>0$ and $c_0=-1/\log P(X_1=A)$. It follows that we need to prove that

$$\lim_{n\to\infty}k^{-1}T_n=\lim_{n\to\infty}k^{-1}U_n=\lim_{n\to\infty}k^{-1}V_n=\lim_{n\to\infty}k^{-1}W_n=A \quad \text{almost surely}.$$

It is straightforward that $T_n\le kA$, and that $V_n\le U_n\le W_n\le kA$. Hence, for $Z_n=T_n, U_n, V_n$ or $W_n$, we have

$$\limsup_{n\to\infty}k^{-1}Z_n\le A \quad \text{almost surely}.$$

In view of the inequality $V_n\le U_n\le W_n$, we need only to prove that, for any $c\in(0,c_0)$, we have

(★) $$\liminf_{n\to\infty}k^{-1}T_n\ge A \quad \text{almost surely}$$

and

(★★) $$\liminf_{n\to\infty}k^{-1}V_n\ge A \quad \text{almost surely}.$$

To prove (★) and (★★), we cut $S_n$ into $[n/k]$ pieces of length $k$ each and let

$$\eta_i = S_{k(i+1)} - S_{ki}, \qquad i = 1, 2, \dots, \left[\frac{n}{k}\right] - 1.$$

The event $\{\eta_i = kA\}$ is equivalent to $X_j = A$ for $j = ki + 1, ki + 2, \dots, k(i + 1)$. It follows evidently that $\{\eta_i = kA\} \Rightarrow V_n \geq kA$ and $T_{ki} \geq \kappa(ki)A$. Hence, we are done if we show that

$$P\left( \bigcap_{[n/2k] \leq i < [n/k]} \{\eta_i < kA\} \text{ i.o.} \right) = 0.$$

We have used here the fact that $\kappa(k[n/2k]) \sim k$ as $n \to \infty$. But we have, for $p = P(X_1 = A)$,

$$\begin{aligned} P\left( \bigcap_{[n/2k] \leq i < [n/k]} \{\eta_i < kA\} \right) &\leq (1 - p^k)^{[n/2k]-1} \leq \exp\left( -\left( \left[\frac{n}{2k}\right] - 1 \right) p^k \right) \\ &\leq e^2 \exp\left( -\frac{n}{2c \log n} \exp\left( \frac{[c \log n]}{-c_0} \right) \right) \\ &\leq e^2 \exp\left( -\frac{n^{1 - c/c_0}}{2c \log n} \right), \end{aligned}$$

which is summable in $n$ when $c \in (0, c_0)$. The result follows by Borel–Cantelli. □

Remark 5. We have just proved that, whenever $0 < c < c_0$, then there exists almost surely an $n_0$ such that $n \geq n_0$ implies $k^{-1}V_n = k^{-1}U_n = k^{-1}W_n = A$. This, in turn, shows that $P(k^{-1}T_n = A \text{ i.o.}) = 1$.

Case (i). We assume now that $t_0 = t^\star(A) < \infty$, $c_0 = c(A) = 1/(At_0 - \log \phi(t_0))$, $A < \infty$.

Lemma 11. *Let $c \in (0, c_0)$. Then, for arbitrary $\varepsilon > 0$, we have*

$$P\left( W_n \geq \left( A + \frac{1}{t_0}\left( \frac{1}{c} - \frac{1}{c_0} \right) + \varepsilon \right) k \quad \textit{i.o.} \right) = 0.$$

Proof. We have

$$P(W_n \geq Ck) \leq n \sum_{i=1}^{k} P(S_i \geq Ck) \leq n \sum_{i=1}^{k} \phi(t_0)^i e^{-Ckt_0} \leq n \frac{\phi(t_0)^{k+1}}{\phi(t_0) - 1} e^{-Ckt_0},$$

by Jensen's inequality. By choosing $C$ so that

$$Ct_0 = \left( A + \frac{1}{t_0}\left( \frac{1}{c} - \frac{1}{c_0} \right) + \varepsilon \right) t_0 = \frac{1}{c} + \log \phi(t_0) + t_0 \varepsilon,$$

we have

$$\begin{aligned} P(W_n \geq Ck) &\leq \phi(t_0)(\phi(t_0) - 1)^{-1} e^{(k+1)/c} e^{-k(t_0\varepsilon + 1/c)} \\ &= \phi(t_0)(\phi(t_0) - 1)^{-1} e^{1/c} e^{-Kt_0\varepsilon}. \end{aligned}$$

But since the right-hand side is summable in $k$ (not in $n$), we can use a subsequence argument as in the proof of Theorem 3 and apply the Borel–Cantelli lemma. This proves Lemma 11. □

LEMMA 12. *Let $c \in (0, c_0)$. Then, for arbitrary $\varepsilon > 0$, we have*

$$P\left(V_n \le \left(A + \frac{1}{t_0}\left(\frac{1}{c} - \frac{1}{c_0}\right) - \varepsilon\right)k \quad i.o.\right) = 0.$$

PROOF. Let $\theta_j(\lambda)$ be defined as in Lemma 6. Let $\varepsilon > 0$ be fixed and put

$$\lambda = A + \frac{1}{t_0}\left(\frac{1}{c} - \frac{1}{c_0}\right) - \varepsilon \quad \text{and} \quad \mu = A + \frac{1}{t_0}\left(\frac{1}{c} - \frac{1}{c_0}\right) + \varepsilon.$$

By the same arguments as in Lemma 6, we have (see, e.g., Remark 1)

$$P(\theta_1(\lambda) > k) = P(S_1 \ge \lambda, S_2 \ge 2\lambda, \dots, S_k \ge k\lambda) \ge k^{-1}P(S_k \ge k\lambda).$$

Next, we need to obtain a lower bound for $P(S_k \ge k\lambda)$. This can be obtained by the following "ghost sample" argument. Let $k'$ be an integer defined by

$$k' + k = 1 + \left[\frac{\mu k}{B}\right], \quad \text{where } B = A - \delta < A \text{ is fixed } (B > 0).$$

Thus, $k'$ increases as $(\mu/B - 1)k$. Since $P(S_{k'} > -2\varepsilon k) \to 1$ by the weak law of large numbers, and $-2\varepsilon k + B(k' + k) \ge -2\varepsilon k + \mu k = \lambda k$, we have

$$\begin{aligned} P(S_k \ge k\lambda) &\sim P(S_k \ge k\lambda)P(S_{k'+k} - S_k > -2\varepsilon k) \\ &\ge P(S_k \ge B(k' + k) - 2\varepsilon k,\, S_{k'+k} - S_k > -2\varepsilon k) \\ &\ge P(S_k \ge B(k' + k) + 2\varepsilon k,\, S_{k'+k} - S_k \ge -2\varepsilon k) \\ &\ge P(S_{k'+k} \ge B(k' + k)). \end{aligned}$$

Now, for any small $\delta > 0$, we have by Theorem C, for any fixed $R > 0$,

$$\begin{aligned} P(S_m \ge (A - \delta)m) &\sim m^{-1/2}\psi(t^{\star})\exp\left(-\frac{m}{c}\right) \\ &\ge Rm^{-1/2}\exp\left(-\frac{m}{c_0}\right), \quad \text{as } m \to \infty, \end{aligned}$$

where $t^{\star} = t^{\star}(A - \delta)$ and $c = c(A - \delta) > c(A) = c_0$.

It follows that there exists a positive constant $v$ such that, as $n \to \infty$,

$$P(S_k \ge k\lambda) \ge vk^{-1/2}\exp\left(-\frac{\mu k}{c_0(A - \delta)}\right),$$

which in turn proves that (by absorbing the small terms) for any $\delta > 0$,

$$P(\theta_1(\lambda) > k) \ge \exp\left(-\frac{\mu k}{c_0(A - \delta)}\right).$$

By this inequality, using the same argument as in the proof of Lemma 10, we get

$$\begin{aligned} P\Big(\sup_{1\le i\le Dn}\theta_i(\lambda)\le k\Big) &\le \exp(-[Dn]P(\theta_1(\lambda)>k))\\ &\le \exp\left(-(Dn-1)\exp\left(-\frac{\mu k}{c_0(A-\delta)}\right)\right)\\ &\le \exp\left(-\exp\left((1+o(1))k\left(\frac{1}{c}-\frac{\mu}{c_0(A-\delta)}\right)\right)\right), \end{aligned}$$

where we have used the fact that $k^{-1}\log(Dn-1)=c^{-1}(1+o(1))$ as $n\to\infty$, for a fixed $D>0$. By the argument of Lemmas 7–10 and Theorem 4, we are done if we can prove that

$$\frac{1}{c}-\frac{\mu}{c_0(A-\delta)}>0.$$

But, by our choice of $\mu=A+t_0^{-1}(c^{-1}-c_0^{-1})+\varepsilon$, noting that $c_0^{-1}=At_0-\log\phi(t_0)$,

$$\begin{aligned} \frac{1}{c}-\frac{\mu}{c_0A} &= \frac{1}{c}-\frac{1}{c_0}\left(1+\frac{1}{t_0A}\left(\frac{1}{c}-\frac{1}{c_0}\right)\right)-\frac{\varepsilon}{c_0A}\\ &= \left(\frac{1}{c}-\frac{1}{c_0}\right)\left(1-\frac{1}{c_0t_0A}\right)-\frac{\varepsilon}{c_0A}\\ &= \left(\frac{1}{c}-\frac{1}{c_0}\right)\frac{\log\phi(t_0)}{At_0}-\frac{\varepsilon}{c_0A}>0, \end{aligned}$$

for all $\varepsilon>0$ small enough. It follows that for such a choice of $\varepsilon$ we can choose also $\delta>0$ small enough such that

$$\frac{1}{c}-\frac{\mu}{c_0(A-\delta)}>0,$$

as sought. This concludes the proof of Lemma 12. □

LEMMA 13. *Let $c\in(0,t_0)$. Then, for arbitrary $\varepsilon>0$, we have*

$$\begin{aligned} &P\left(T_n\le\left(A+\frac{1}{t_0}\left(\frac{1}{c}-\frac{1}{c_0}\right)-\varepsilon\right)k \quad i.o.\right)\\ &\qquad = P\left(T_n\ge\left(A+\frac{1}{t_0}\left(\frac{1}{c}-\frac{1}{c_0}\right)+\varepsilon\right)k \quad i.o.\right)=0. \end{aligned}$$

PROOF. It follows along the same lines as Lemmas 11 and 12, using the techniques of Deheuvels, Devroye and Lynch (1986) [see also Deheuvels and Devroye (1983)]. Details will therefore be omitted. □

REMARK 6. The full form of the Erdös–Rényi and Shepp theorems is obtained by combining Theorem 5 with Theorem A. We note that the result corresponding to case (i) of Theorem B was overlooked by several authors.

**6. The limiting distribution of the Erdös–Rényi statistic.** In this section, we assume throughout that $c = c(\alpha) \in (c_0, \infty)$. We will limit ourselves to the evaluation of the limiting distribution of $U_n$ as $n \to \infty$, noting that our methods can be extended to the other random variables in the study. Throughout this section, we assume that the distribution of $X_1$ is nonlattice. Our main result is as follows.

THEOREM 6. *There exists a constant* $\Delta \in (0,1]$, *depending upon* $\alpha$ *and the distribution of* $X_1$, *such that, for all* $y \in \mathbb{R}$,

$$\lim_{n\to\infty} P\big((U_n - \alpha k)t^\star + \tfrac{1}{2}\log k - \log\{n\Delta\psi(t^\star)e^{-k/c}\} \le y\big) = \exp(-e^{-y}).$$

Note for further use that $ne^{-k/c}$ oscillates between 1 and $e^{1/c}$ (this is due to the fact that $c\log n - [c\log n]$ fluctuates between 0 and 1). The proof is captured in the following sequence of lemmas.

LEMMA 14. *For all* $y \in \mathbb{R}$, *we have, as* $n \to \infty$,

$$P\big(U_n \le \alpha k - (2t^\star)^{-1}\log k + y/t^\star\big) \ge \exp\big(-ne^{-k/c}\{\psi(t^\star) + o(1)\}e^{-y}\big).$$

PROOF. By an association inequality of Deheuvels and Devroye (1984), Lemma 10, and noting that for $0 < p < 1$, $(1-p)^n \ge \exp(-np/(1-p))$, we have

$$\begin{aligned} P\Big(\max_{0\le i\le n-k}(S_{i+k} - S_i) \le x\Big) &\ge P^n(S_k \le x) \\ &\ge \exp\big(-nP(S_k > x)/(1 - P(S_k > x))\big). \end{aligned}$$

Let $x = \alpha k - (2t^\star)^{-1}\log k + y/t^\star$. By Corollary 1, we have

$$P(S_k > x) \sim \psi(t^\star)e^{-k/c}e^{-y} \to 0, \quad \text{as } n \to \infty,$$

which suffices for proof of Lemma 14. □

LEMMA 15. *Let* $x = \alpha k - (2t^\star)^{-1}\log k + y/t^\star$. *There exists a constant* $q \in (0,1)$ *and an* $n_0 < \infty$, *such that, for all* $n \ge n_0$, *we have, for* $1 \le j \le k$,

$$P\big(S_k > x, S_{k+j} - S_j > x\big) \le 2q^j P(S_k > x).$$

PROOF. We start with the inequality [see Deheuvels, Devroye and Lynch (1986), Lemma 4]

$$P\big(S_k > x, S_{k+j} - S_j > x\big) \le P\big(S_{k-j} > r\big) + P\big(S_j > x - r\big)P\big(S_k > x\big).$$

Next, we choose an arbitrary $\beta \in (0, (t^\star)^{-1}\log\phi(t^\star))$, and let, in the inequality above, $r = x - \beta j$. By Jensen's (or Markov's) inequality,

$$P\big(S_j > x - r\big) = P\big(S_j > \beta j\big) \le e^{-j/c'},$$

where $c' = c(\beta)$ is defined via the relation $e^{-1/c'} = \inf_{t>0}\phi(t)e^{-t\beta} < 1$.

By Corollary 1, we have, uniformly in $1 \le j \le k^{1/3}$, as $n \to \infty$,

$$P(S_{k-j} > r) \sim \psi(t^\star)e^{-(k-j)/c}e^{-y}e^{-(\alpha-\beta)jt^\star}\left(\frac{k}{k-j}\right)^{1/2} \sim P(S_k > x)e^{j/c-(\alpha-\beta)jt^\star}$$
$$= P(S_k > x)\exp(j\{\beta t^\star - \log\phi(t^\star)\}).$$

Also, by Jensen's inequality, we have, uniformly in $k^{1/3} \le j \le k$, as $n \to \infty$,

$$P(S_{k-j} > r) \le \phi(t^\star)^{k-j}e^{-t^\star r} = \phi(t^\star)^{-j}e^{-k/c}k^{1/2}e^{-y}e^{j\beta t^\star}$$
$$\le P(S_k > x)\{(1+o(1))/\psi(t^\star)\}k^{1/2}e^{j\{\beta t^\star - \log\phi(t^\star)\}}$$
$$\le P(S_k > x)\{(1+o(1))/\psi(t^\star)\}j^{3/2}e^{j\{\beta t^\star - \log\phi(t^\star)\}}$$
$$\le P(S_k > x)\exp\left(\frac{j}{1+\varepsilon}\{\beta t^\star - \log\phi(t^\star)\}\right),$$

where $\varepsilon > 0$ is fixed. It follows that there exists $n_0$ (depending on $\varepsilon > 0$) such that, for $n \ge n_0$,

$$P(S_k > x, S_{k+j} - S_j > x) \le 2P(S_k > x)q^j,$$

where $q = \max\{e^{-1/c'}, \exp((1/(1+\varepsilon))\{\beta t^\star - \log\phi(t^\star)\})\} < 1$ as desired. □

In the sequel, it will be convenient to let $x = \alpha k - (2t^\star)^{-1}\log k + y/t^\star$, and $A_i = \{S_{i+k} - S_i > x\}$, $i = 0,\dots,n-k$. Note that $\{U_n > x\} = \bigcup_{i=0}^{n-k}A_i$, and that, for $1 \le j \le k$,

$$P(A_i \cap A_{i+j}) = P(S_k > x, S_{k+j} - S_j > x) \quad \text{and} \quad P(A_i) = P(S_k > x).$$

Lemma 16. *Let $\varepsilon > 0$ be fixed. There exists an integer $I \ge 1$ such that, for any integer $r \ge 1$, there exists an $n_0$ such that $n \ge n_0$ implies*

$$\frac{rk(1-\varepsilon)}{I}P\left(\bigcup_{i=1}^{I}A_i\right) \le P\left(\bigcup_{i=1}^{rk}A_i\right) \le \frac{rk(1+\varepsilon)}{I}P\left(\bigcup_{i=1}^{I}A_i\right).$$

Proof. The right-hand side inequality is straightforward by Bonferroni if we note that

$$\bigcup_{i=1}^{rk}A_i \subset \bigcup_{j=1}^{N+1}\left\{\bigcup_{i=1}^{I}A_{(j-1)I+i}\right\}, \quad \text{where} \quad N = [rk/I].$$

For the left-hand side, we first choose an integer $J = I + K \ge I$, and consider the events

$$B_j = \bigcup_{i=1}^{I}A_{(j-1)J+i}, \qquad j = 1,2,\dots,M = [rk/J].$$

By Bonferroni, we have

$$P\left(\bigcup_{i=1}^{rk}A_i\right) \ge P\left(\bigcup_{j=1}^{M}B_j\right) \ge MP(B_1) - \sum_{1\le j<l\le M}P(B_j \cap B_l).$$

Next by Lemma 15, we see that, for $n \geq n_0$, if $|j - l|K \leq k$,

$$P(B_j \cap B_l) \leq 2I^2 P(A_0) q^{|j-l|K},$$

and hence

$$\sum_{1 \leq j < l \leq M} P(B_j \cap B_l) \leq 2MI^2 P(A_0) \sum_{i=1}^{\infty} q^{iK}$$

$$-2I^2 P(A_0) \left( \frac{q^k}{1 - q^k} \right)^2 q^k (1 - q^{(M-1)k}),$$

while $MP(B_1) \geq MP(A_0)$.

Choose now $K = [\varepsilon I/4]$, and observe that, for $I \to \infty$, $I^2 q^K/(1 - q^K) \to 0$, while $J/I \to 1 + \varepsilon/4$. This enables us to select an $I$ such that, uniformly over $r$ with $rkP(A_0) \leq \varepsilon/4$, we have

$$P\left( \bigcup_{i=1}^{rk} A_i \right) \geq MP(B_1)\left(1 - \frac{\varepsilon}{2}\right) \geq \frac{rk(1 - \varepsilon)}{I} P(B_1), \quad \text{as sought.}$$

Note for further use that our construction implies that any choice of $I$ exceeding the value previously chosen will also satisfy the conclusions of Lemma 16. □

LEMMA 17. *Let* $A_i = \{S_{i+k} - S_i > \alpha k - (2t^\star)^{-1} \log k + y/t^\star\}$ *and* $C_i = \{S_{i+k} - S_i > \alpha k - (2t^\star)^{-1} \log k\}$, $i = 0, 1, \dots, n - k$. *For any fixed integer* $I \geq 1$ *and* $y \in \mathbb{R}$, *we have*

$$\lim_{n \to \infty} P\left( \bigcup_{i=1}^{I} A_i \right) \Big/ P\left( \bigcup_{i=1}^{I} C_i \right) = e^{-y}.$$

PROOF. By Corollary 1, for all $i \geq 0$, $P(A_i) = P(A_0) \sim P(C_0)e^{-y} = P(C_i)e^{-y}$ as $n \to \infty$. Also, by the inclusion–exclusion principle,

$$P\left( \bigcup_{i=1}^{I} A_i \right) = \sum_{i=1}^{I} P(A_i) - \sum_{1 \leq i < j \leq I} P(A_i \cap A_j) + \cdots$$

$$+(-1)^{I+1} P(A_1 \cap \cdots \cap A_I).$$

Since $I$ is constant and $P(A_0) \leq P(\bigcup_{i=1}^{I} A_i) \leq IP(A_0)$, the proof will follow if we show that, for any $2 \leq j \leq I$ and $1 \leq i_1 < \cdots < i_j \leq I$, we have, as $n \to \infty$,

$$P(A_{i_1} \cap \cdots \cap A_{i_j}) = (1 + o(1)) P(C_{i_1} \cap \cdots \cap C_{i_j}) e^{-y} + o(P(A_0)).$$

For $m = 0, 1$ or $2$, let $\Sigma_m = \{$integers in the interval $J_m\}$, where $J_0 = (i_j, i_1 + k]$, $J_1 = (i_1, i_j]$ and $J_2 = (i_1 + k, i_j + k]$. For $n$ large enough, $\#\Sigma_0 = k - r = k - (i_j - i_1)$, $\#\Sigma_1 = \#\Sigma_2 = r$, where $r \geq 1$ is independent of $n$ and $\#\Sigma_m$ denotes the cardinality of $\Sigma_m$. For $m = 0$, $1$ or $2$, let $T_m = \sum_{l \in \Sigma_m} X_l$. It is straightforward that, for $n$ large enough,

$$\theta(s) = P\big(A_{i_1} \cap \cdots \cap A_{i_j} | T_0 = x - s\big) = P\big(C_{i_1} \cap \cdots \cap C_{i_j} | T_0 = x - s - y/t^\star\big)$$

is nonincreasing and right-continuous in $s$. For instance, for $j = 2$, we have exactly $\theta(s) = P(T_1 > s)P(T_2 > s)$. In general, we have as an upper bound

$$\theta(s) \le P(T_1 > s)P(T_2 > s) = P(S_r > s)^2.$$

Let $m = k^{1/3}$. By integrating with respect to $s$, we have

$$\int_{-\infty}^{+\infty} \theta(s)\, dP(T_0 \ge x - s) = \int_{-m}^{m} \theta(s)\, dP(T_0 \ge x - s) + \int_{|t|>m} \theta(s)\, dP(T_0 \ge x - s) = I_1 + I_2.$$

By integrating $I_1$ by parts, we obtain

$$I_1 = \theta(m)P(T_0 \ge x - m) - \theta(-m)P(T_0 \ge x + m) + \int_{-m}^{m} P(T_0 \ge x - s)\, d\theta(s) = I_{11} - I_{12} + I_{13}.$$

Next, we see that, as $n \to \infty$,

$$I_{11} \le P(S_{k-r} \ge x - m)P(S_r > m)^2 \le P(S_k > x)P(S_r > m) = o\big(P(S_k > x)\big) = o\big(P(A_0)\big).$$

Likewise, by Corollary 1, as $n \to \infty$,

$$I_{12} \le P(S_{k-r} \ge x + m) = o\big(P(A_0)\big).$$

Using the same arguments, we get

$$I_2 \le P(S_{k-r} \ge x - m)P(S_r > m)^2 + P(S_{k-r} > x + m) = o\big(P(A_0)\big), \quad \text{as} \quad n \to \infty.$$

For the remaining term, we use again Corollary 1 to show that, uniformly in $|s| \le m = k^{1/3}$, we have $P(T_0 \ge x - s) \sim P(T_0 \ge x - s - y/t^\star)e^{-y}$. By using once again the same argument in reversed order, this shows that, as $n \to \infty$,

$$\begin{aligned} P\big(A_{i_1} \cap \cdots \cap A_{i_j}\big) &= (1 + o(1)) \int_{-m}^{m} P(T_0 \ge x - s)\, d\theta(s) + o\big(P(A_0)\big) \\ &= o\big(P(A_0)\big) + (1 + o(1))e^{-y} \\ &\quad \times \int_{-m}^{m} P(T_0 \ge x - s - y/t^\star)\, d\theta(s) \\ &= (1 + o(1))e^{-y} P\big(C_{i_1} \cap \cdots \cap C_{i_j}\big) + o\big(P(A_0)\big), \end{aligned}$$

as sought. □

LEMMA 18. *There exists a constant $\Gamma > 0$, depending upon the distribution of $X_1$ only, such that, for all $y \in \mathbb{R}$, we have, as $n \to \infty$,*

$$P\big(U_n \le \alpha k - (2t^\star)^{-1}\log k + y/t^\star\big) \le \exp\big(-ne^{-k/c}\Gamma\{\psi(t^\star) + o(1)\}e^{-y}\big).$$

**Proof.** Let, as usual, $A_i = \{S_{i+k} - S_i > x\}$ and $x = \alpha k - (2t^\star)^{-1}\log k + y/t^\star$. By Lemma 15 and Corollary 1, we have

(i) $$P(A_i) = P(A_0) = p \sim \psi(t^\star)e^{-k/c}e^{-y};$$

(ii) $$P(A_i \cap A_j) = p^2, \qquad |i-j| > k;$$

(iii) $$P(A_i \cap A_j) \le 2pq^{|i-j|}, \qquad 1 \le |i-j| \le k,$$

where $q \in (0,1)$ is a constant.

Let $m = [(n-k)/2k]$, and $B_j = \bigcap_{l=1}^{k} A^c_{(j-1)2k+l}$, $1 \le j \le m$. Then, by the independence of the $B_j$'s,

$$P(U_n \le x) = P\left(\bigcap_{i=0}^{n-k} A_i^c\right) \le P\left(\bigcap_{j=1}^{m} B_j\right) = \prod_{j=1}^{m} P(B_j) \le \exp\left(-\sum_{j=1}^{m} P(B_j^c)\right).$$

Also, by the Chung–Erdös (1952) inequality and our assumptions, for any $1 \le j \le m$, we have

$$P(B_j^c) \ge \frac{(kp)^2}{kp + \Sigma_{1\le i\ne j\le k} P(A_i \cap A_j)} \ge \frac{(kp)^2}{kp + (kp)^2 + 4kp\Sigma_{l\ge 1} q^l}$$
$$= \frac{kp(1-q)}{1-q+4q+kp} \sim \frac{kp(1-q)}{1+3q}, \quad \text{as } n \to \infty \text{ since } kp \to 0.$$

Since $m \ge (n-k)/2k - 1 \sim n/2$, we obtain the stated result without further work, together with the lower bound $\Gamma \ge \frac{1}{2}(1-q)/(1+3q)$. □

**Proof of Theorem 6.** Let $0 < \varepsilon < \frac{1}{4}$ be fixed. By Lemma 16, there exists an integer $I = I_\varepsilon \ge 1$, such that, for any integer $r \ge 1$, there exists an $n_{\varepsilon,r}$ such that $n \ge n_{\varepsilon,r}$ implies

$$\frac{rk(1-\varepsilon)}{I} P\left(\bigcup_{i=1}^{I} A_i\right) \le P\left(\bigcup_{i=1}^{rk} A_i\right) \le \frac{rk(1+\varepsilon)}{I} P\left(\bigcup_{i=1}^{I} A_i\right).$$

Let $r \ge 1$ be fixed, and let $m = [(n-k)/(r+1)k]$. Consider the events

$$D_j = \bigcup_{l=1}^{rk} A_{(j-1)(r+1)k+l} \quad \text{and} \quad E_j = \bigcup_{l=1}^{k} A_{(j-1)(r+1)+rk+l}, \qquad j = 1,2,\dots.$$

The events $\{D_j,\ j \ge 1\}$ (resp. $\{E_j,\ j \ge 1\}$) are independent. It follows that

$$P(U_n \le x) = P\left(\bigcap_{i=0}^{n-k} A_i^c\right) \le P\left(\bigcap_{j=1}^{m} D_j^c\right) = \prod_{j=1}^{m} P(D_j^c)$$
$$\le \exp\left(-\frac{mrk(1-\varepsilon)}{I} P\left(\bigcup_{i=1}^{I} A_i\right)\right).$$

Note that $mrk \sim nr/(r+1)$ as $n \to \infty$. Choose $r$ such that $mrk(1-\varepsilon) \ge (1-2\varepsilon)n$. Lemma 17 implies the existence of $n'_\varepsilon$ (depending upon $\varepsilon$ through $I = I_\varepsilon$) such that

$$(1-2\varepsilon)P\left(\bigcup_{i=1}^{I} A_i\right) \ge (1-3\varepsilon)e^{-y} P\left(\bigcup_{i=1}^{I} C_i\right), \quad \text{for } n \ge n'_\varepsilon.$$

By all this, for any $n \geq \max(n'_\varepsilon, n_{\varepsilon,r})$,

$$P(U_n \leq x) \leq \exp\left(-(1-3\varepsilon)\frac{ne^{-y}}{I}P\left(\bigcup_{i=1}^{I} C_i\right)\right). \tag{6.1}$$

By a similar argument,

$$\begin{aligned} P(U_n \leq x) &\geq P\left(\bigcap_{j=1}^{m+2} D_j^c\right) - P\left(\bigcup_{j=1}^{m+2} E_j\right) \\ &\geq (1 - P(D_1))^{m+2} - (m+2)kP(A_0). \end{aligned} \tag{6.2}$$

We now make use of Lemma 14 which (jointly with Corollary 1) implies the existence of an absolute constant $\Omega > 0$ such that, for $n$ large enough, we have

$$P(U_n \leq x) \geq \exp(-n\Omega P(A_0)).$$

Using the fact that $(m+2)k \sim n/(r+1)$ and that $nP(A_0) \sim \psi(t^\star)ne^{-k/c}e^{-y} \leq \psi(t^\star)e^{1/c}e^{-y}$ enables us to choose $r$ so great that, uniformly over $|y| \leq y_0$ fixed, we have from (6.2) that

$$\begin{aligned} &P(U_n \leq x) + (m+2)kP(A_0) \\ &\quad \leq P(U_n \leq x)\{1 + (m+2)kP(A_0)\exp(n\Omega P(A_0))\} \\ &\quad \leq P(U_n \leq x)/(1-\varepsilon), \end{aligned}$$

which in turn gives

$$(1 - P(D_1))^{m+2} \leq P(U_n \leq x) + (m+2)kP(A_0) \leq P(U_n \leq x)/(1-\varepsilon).$$

Consequently,

$$\begin{aligned} P(U_n \leq x) &\geq (1-\varepsilon)(1-P(D_1))^{m+2} \\ &\geq (1-\varepsilon)\exp(-(m+2)P(D_1)/(1-P(D_1))). \end{aligned} \tag{6.3}$$

Recall that $P(D_1) = P(\bigcup_{i=1}^{rk} A_i) \leq (rk/n)(nP(A_0)) = O(k/n)$ uniformly over $|y| \leq y_0$. It follows that $(m+2)P(D_1)/(1-P(D_1)) = (m+2)P(D_1)(1 + O(k/n))$ as $n \to \infty$. By the same arguments as above, we can therefore show the existence of $n''_\varepsilon$ such that $n \geq n''_\varepsilon$ implies, for all $r$ sufficiently large,

$$\begin{aligned} (m+2)P(D_1)/(1-P(D_1)) &\leq (1+\varepsilon)\frac{n}{k(r+1)}P\left(\bigcup_{i=1}^{rk} A_i\right) \\ &\leq (1+2\varepsilon)\frac{n}{I}P\left(\bigcup_{i=1}^{I} A_i\right) \leq (1+3\varepsilon)\frac{ne^{-y}}{I}P\left(\bigcup_{i=1}^{I} C_i\right). \end{aligned}$$

Combining this with (6.1) and (6.3) we get

$$\begin{aligned} (1-\varepsilon)\exp\left(-(1+3\varepsilon)\frac{ne^{-y}}{I}P\left(\bigcup_{i=1}^{I} C_i\right)\right) &\leq P(U_n \leq x) \\ &\leq \exp\left(-(1-3\varepsilon)\frac{ne^{-y}}{I}P\left(\bigcup_{i=1}^{I} C_i\right)\right). \end{aligned}$$

Observe that $P(\bigcup_{i=1}^{I} C_i)$ is independent of $y$. We will now prove that, for a fixed $I$, the limit $\Delta_I = \lim_{n\to\infty} P(\bigcup_{i=1}^{I} C_i)/\{IP(C_0)\}$ exists. The proof of this result is similar to that of Lemma 17 and consists in showing that, for any $2 \le j \le I$ and $1 \le i_1 < \cdots < i_j \le I$, the limit $\lim_{n\to\infty} P(C_{i_1} \cap \cdots \cap C_{i_j})/P(C_0)$ exists. For this, we use the notation of the proof of Lemma 17 with $y = 0$ (so that $A_i = C_i$, $i = 0, 1, \dots$), and observe that there exists a fixed integer $r$ and a bounded nondecreasing function $\theta(\cdot)$ independent of $n$ such that

$$P\left(C_{i_1} \cap \cdots \cap C_{i_j}\right) = (1 + o(1)) \int_{-k^{1/3}}^{k^{1/3}} P(S_{k-r} \ge x - s)\, d\theta(s); \quad \text{as } n \to \infty.$$

By Corollary 1, $P(S_{k-r} \ge x - s) = (1 + o(1))P(S_k \ge x)\exp(r/c - \alpha r t^\star + s t^\star)$, where the "$o(1)$" term is uniform over $|s| \le k^{1/3}$. It is now straightforward that

$$\lim_{n\to\infty} P\left(C_{i_1} \cap \cdots \cap C_{i_j}\right)/P(C_0) = \exp\left(\frac{r}{c} - \alpha r t^\star\right) \int_{-\infty}^{\infty} e^{s t^\star}\, d\theta(s).$$

Note that $P(C_0) \le P(\bigcup_{i=1}^{I} C_i) \le IP(C_0)$, and hence that $1/I \le \Delta_I \le 1$. By all this, for all $n$ sufficiently large, we have

$$\begin{aligned}(1 - \varepsilon)\exp(-(1 + 4\varepsilon)\Delta_I n P(C_0) e^{-y}) &\le P(U_n \le x) \\ &\le \exp(-(1 - 4\varepsilon)\Delta_I n P(C_0) e^{-y}).\end{aligned}$$

Using again Corollary 1, we see that $nP(C_0) \sim \psi(t^\star) n e^{-k/c}$ as $n \to \infty$. Finally, the proof of Theorem 7 is completed by the remark that, by Lemmas 14 and 18,

$$\Gamma \le \liminf_{I\to\infty} \Delta_I \le \limsup_{I\to\infty} \Delta_I \le 1,$$

which implies the existence of a convergent subsequence $\Delta_{I_l} \to \Delta \in [\Gamma, 1]$ as $l \to \infty$. Since $\varepsilon > 0$ (resp. $I$) can be chosen as small (resp. as great) as desired, the proof follows. □

REMARK 7. In the case where the distribution of $X_1$ is lattice with span $H$, Theorem 6 is invalid if $y$ is not restricted to be such that

$$\alpha k - (2t^\star)^{-1}\left(((\log k)/2) + \log\{n\Delta\psi(t^\star)e^{-k/c}\} + y\right)$$

belongs to the subgroup of $\mathbb{R}$ generated by the support of $X_1$. We will not state here the corresponding result, noting that our methods enable us to treat this case also.

**7. Applications.** We limit ourselves in this section to the following simple examples.

EXAMPLE 1 (The Wiener process). Révész (1982), followed by Ortega and Wschebor (1984), have studied the random variable

$$\xi(T) = \sup_{0 \le t \le T-b} \{W(b + t) - W(t)\},$$

where $\{W(t),\ t \ge 0\}$ is a Wiener process, and where $b = b_T$ is a nondecreasing function of $T$. We shall be concerned here with the case where

$$b_T = c \log T,$$

where $c > 0$ is a given constant. Under these conditions, we have

THEOREM D (Révész–Ortega–Wschebor).

(i) $\limsup_{T\to\infty} \left(\xi(T) - \sqrt{2c}\log T\right)/\log\log T = \sqrt{c/8}$ *almost surely*;

(ii) $\liminf_{T\to\infty} \left(\xi(T) - \sqrt{2c}\log T\right)/\log\log T = -\sqrt{c/8}$ *almost surely*.

PROOF. (i) follows from Theorem 1 in Ortega and Wschebor (1984), while (ii) follows from Theorem 2.1 in Révész (1982). □

It is worthwhile to compare Theorem D with the following corollary of Theorem A.

COROLLARY 3. *Let $X_1, X_2, \ldots$ be independent standard normal random variables, and let $k$ be $[c\log n]$, $c > 0$. Then*

(i) $\limsup_{n\to\infty} \left(U_n - \sqrt{2c}\log n\right)/\log\log n = \sqrt{c/8}$ *almost surely*;

(ii) $\liminf_{n\to\infty} \left(U_n - \sqrt{2c}\log n\right)\log\log n = -\sqrt{c/8}$ *almost surely*;

(iii) $\left(U_n - \sqrt{2c}\log n\right)/\log\log n \to -\sqrt{c/8}$ *in probability*.

A comparison between Theorem D and Corollary 3 shows that the results are in agreement. We note here that without loss of generality one can put $X_n = W(n+1) - W(n)$. This gives $U_n \leq \xi(n)$. We note also that similar results could be obtained for the modulus of continuity of $W(\cdot)$.

EXAMPLE 2 (Spacings). Let $U_1, U_2, \ldots$ be a sequence of independent and uniformly distributed on $(0,1)$ random variables. Denote the order statistics of $U_1, \ldots, U_n$ by

$$0 = U_{0,n} < U_{1,n} < \cdots < U_{n,n} < 1 = U_{n+1,n}.$$

For any $k = 1, 2, \ldots, n$, the maximal $k$th spacing $K_n = K_n(k)$ is defined as

$$K_n = \max_{0\leq i\leq n+1-k} \{U_{i+k,n} - U_{k,n}\}.$$

Note here that $K_n$ corresponds to the modulus of continuity of the empirical quantile function of $U_1, \ldots, U_n$. The limiting behavior of $K_n$ when $k = [c\log n]$ can be deduced from the fact that the random variables $U_{i+1,n} - U_{i,n}$, $i = 0, \ldots, n$ are identical in distribution with $X_i/\sum_{l=0}^{n} X_l$, where $X_0, X_1, \ldots, X_n$ is a sequence of independent and exponentially distributed random variables [see, e.g., Deheuvels and Devroye (1984)]. As a simple corollary of Theorem A, we have

COROLLARY 4.

$$\frac{nK_n - (1+a)c\log n}{\log\log n} \to -\frac{1}{2}\frac{a+1}{a} \quad \textit{in probability},$$

*where $a > 0$ is arbitrary and $c > 0$ is related to $a$ via the equation*

$$\exp\left(-\frac{1}{c}\right) = (1 + a)e^{-a}.$$

PROOF. By the central limit theorem, $n^{-1}\sum_{l=0}^{n} X_l - 1 = O_P(n^{-1/2})$. The result follows by applying Theorem A to the sequence $X_0, \dots, X_n$. □

Strong versions of this result are detailed in Deheuvels and Devroye (1984).

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