Bézier Curves

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September 11, 2002

Quadratic Bézier curves

Let P_0, P_1, P_2 be distinct points. If points U, V, B divide the line segments P_0P_1, P_1P_2, UV by an equal ratio, then B moves on a quadratic Bézier curve if U moves about P_0P_1.
Thus for some real number t,

$$\begin{aligned} U - P_0 &= t \cdot (P_1 - P_0), \\ V - P_1 &= t \cdot (P_2 - P_1), \\ B - U &= t \cdot (V - U), \end{aligned}$$

where $0 \leq t \leq 1$.

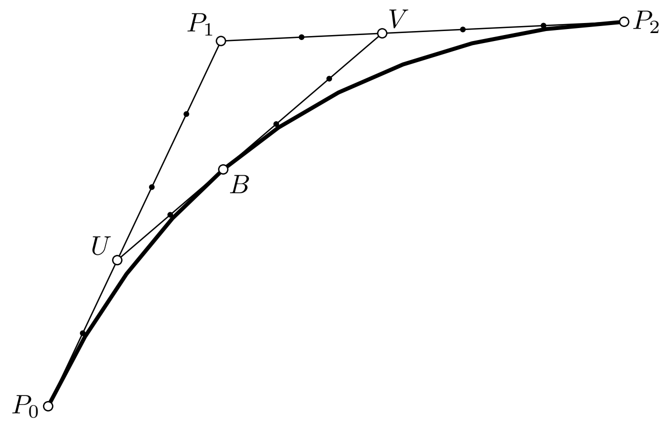

Fig. 1: Quadratic Bézier curve

From the above equations, it follows that $U = P_0 + t \cdot (P_1 - P_0) = P_0 + t \cdot P_1 - t \cdot P_0$, therefore

$$U = (1-t) \cdot P_0 + t \cdot P_1. \tag{1}$$

We equally find $V = (1-t) \cdot P_1 + t \cdot P_2$ and $B = (1-t) \cdot U + t \cdot V$. Substitution yields $B = (1-t) \cdot ((1-t) \cdot P_0 + t \cdot P_1) + t \cdot (V = (1-t) \cdot P_1 + t \cdot P_2)$, and after expansion,

$$B = (1-t)^2 \cdot P_0 + 2(1-t)t \cdot P_1 + t^2 \cdot P_2. \tag{2}$$

Cubic Bézier curves

Let P_0, P_1, P_2, P_3 be distinct points, and let the points $P_4, \ldots, P_8, B$ divide their respective line segments by an equal ratio:

$$\begin{aligned} P_4 - P_0 &= t \cdot (P_1 - P_0), \\ P_5 - P_1 &= t \cdot (P_2 - P_1), \\ P_6 - P_2 &= t \cdot (P_3 - P_2), \\ P_7 - P_4 &= t \cdot (P_5 - P_4), \\ P_8 - P_5 &= t \cdot (P_6 - P_5), \\ B - P_7 &= t \cdot (P_8 - P_7), \end{aligned}$$

where $0 \leq t \leq 1$. (In fig. 2 as well as in fig. 1, $t = 0.4$.)

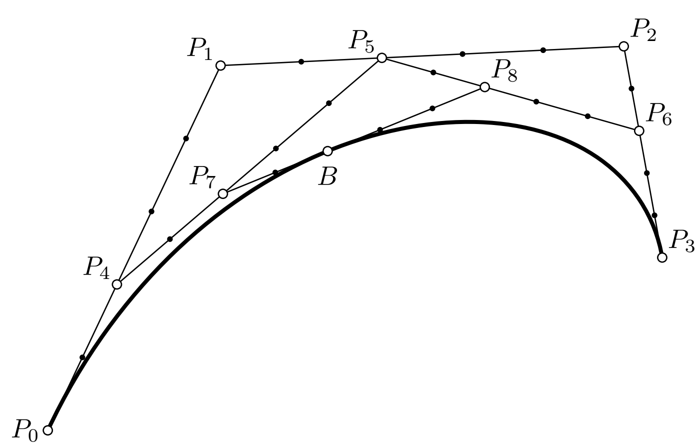

Fig. 2: Cubic Bézier curve

From (2), we get

$$\begin{cases} P_7 = (1-t)^2 \cdot P_0 + 2(1-t)t \cdot P_1 + t^2 \cdot P_2, \\ P_8 = (1-t)^2 \cdot P_1 + 2(1-t)t \cdot P_2 + t^2 \cdot P_3. \end{cases}$$

From (1), $B = (1-t) \cdot P_7 + t \cdot P_8 = (1-t) \cdot [(1-t)^2 \cdot P_0 + 2(1-t)t \cdot P_1 + t^2 \cdot P_2] + t \cdot [(1-t)^2 \cdot P_1 + 2(1-t)t \cdot P_2 + t^2 \cdot P_3]$ which, after expansion, yields

$$B = (1-t)^3 \cdot P_0 + 3(1-t)^2 t \cdot P_1 + 3(1-t)t^2 \cdot P_2 + t^3 \cdot P3. \tag{3}$$

In fig. 2, if P_4 moves about the segment P_0P_1, then P_7 moves on the quadratic Bézier curve determined by points P_0, P_1, P_2, while P_8 moves on the quadratic Bézier curve determined by points P_1, P_2, P_3.

Bézier curves of arbitrary order

For distinct points P_0, P_1, … P_n, the Bézier curve of order n ($n = 0, 1, 2, \ldots$) can be recursively defined by

$$\begin{cases} B_0(t, P_0) &:= P_0, \\ B_n(t, P_0, \ldots, P_n) &:= (1-t)\cdot B_{n-1}(t, P_0, \ldots, P_{n-1}) \\ & \quad +t\cdot B_{n-1}(t, P_1, \ldots, P_n) \qquad (n > 0), \end{cases} \tag{4}$$

where $0 \le t \le 1$.

Theorem 1 (Bézier curves of order 1) *For points P_0, P_1, the Bézier curve of order 1 is given by the equation*

$$B_1(t, P_0, P_1) = (1-t)\cdot P_0 + t\cdot P_1.$$

PROOF: By the above definition, $B_1(t, P_0, P_1) = (1-t)\cdot B_0(t, P_0) + t\cdot B_0(t, P_1) = (1-t)\cdot P_0 + t\cdot P_1$. $\dashv$

Theorem 2 *For any non-negative integer n, the Bézier curve of order n is given by the equation*

$$B_n(t, P_0, \ldots, P_n) = \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k \cdot P_k.$$

PROOF: By induction on n. For $n = 0$, the theorem claims

$$B_0(t, P_0) = \sum_{k=0}^{0} \binom{n}{0} (1-t)^0 t^0 \cdot P_0 = P_0,$$

which is correct by the first part of the definition.

For $n > 0$, we have $B_n(t, P_0, \ldots, P_n) = (1-t)\cdot B_{n-1}(t, P_0, \ldots, P_{n-1}) + t\cdot B_{n-1}(t, P_1, \ldots, P_n)$ by definition. By induction hypothesis,

$$B_n(t, P_0, \ldots, P_n) =$$
$$(1-t)\left[\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k \cdot P_k\right] + t\left[\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k \cdot P_{k+1}\right].$$

We get

$$(1-t)\left[\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k \cdot P_k\right] = (1-t)^n \cdot P_0 + \sum_{k=1}^{n-1} \binom{n-1}{k} (1-t)^{n-k} t^k \cdot P_k,$$

$$t\left[\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k \cdot P_{k+1}\right] = \sum_{k=0}^{n-2} \binom{n-1}{k} (1-t)^{n-1-k} t^{k+1} \cdot P_{k+1} + t^n \cdot P_n$$

$$= \sum_{k=1}^{n-1} \binom{n-1}{k-1} (1-t)^{n-k} t^k \cdot P_k + t^n \cdot P_n.$$

Substituting the last two results, we get

$$B_n(t, P_0, \ldots, P_n) = (1-t)^n \cdot P_0 + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] (1-t)^{n-k} t^k \cdot P_k + t^n \cdot P_n$$

$$= \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k \cdot P_k, \qquad \text{as } \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

by the fundamental law of the binomial coefficients. $\dashv$