RANDOM VARIATE GENERATION FOR THE DIGAMMA AND TRIGAMMA DISTRIBUTIONS

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ABSTRACT. We derive uniformly fast random variate generators for Sibuya's digamma and trigamma families. Some of these generators are based upon the close resemblance between these distributions and selected generalized hypergeometric distributions. The generators can also be used for the discrete stable distribution, the Yule distribution, Mizutani's distribution and the Waring distribution.

KEYWORDS AND PHRASES. Digamma distribution. Random variate generation. Trigamma distribution. Probability inequalities. Rejection method. Expected time complexity. Generalized hypergeometric family.

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Introduction.

The object of this paper is to discuss efficient methods for the generation of random variates for the digamma and trigamma distributions introduced by Sibuya (1979). In the process, we obtain efficient algorithms for a variety of distributions including the discrete stable distribution. The key to these distributions is the versatile generalized hypergeometric family of type B3, or GHgB3, which is a discrete distribution on the nonnegative integers defined by

$$p_n = \frac{\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)\Gamma(c)} \frac{(a)_n (b)_n}{n!(a+b+c)_n} \ (n \geq 0),$$

where $(a)_n$ is Pochhammer's symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\dots(a+n-1) & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

and a, b, c are positive parameters of the distribution. Other names for the distribution include the inverse Pólya-Eggenberger distribution (Johnson and Kotz, 1982), the generalized Waring distribution (Irwin, 1965, 1968, 1975), and the negative binomial beta distribution. Special cases include the Yule distribution (Simon, 1954, 1960) obtained for $a = b = 1$, the Mizutani distribution (Mizutani, 1953) obtained for $b = c = 1$ and the Waring distribution, obtained for $a = 1$ or $b = 1$. For $a = 1$, $b = k - r + 1$, $c = r - 1$, with $r \geq k \geq 2$ all integer, we obtain the factorial distribution (also called beta-Pascal distribution) of Marlow (1965). Sibuya (1979) introduced a simple rejection method for generating digamma random variates based upon the closeness to the GHgB3 distribution. His algorithm is not uniformly fast over the entire parameter space. In this paper, we present a host of digamma generators that halt in expected time uniformly bounded over all choices of the parameters.

The GHgB3 distribution.

From Sibuya (1979), Shimizu (1968) and Sibuya and Shimizu (1981) we recall the following:

A. The distribution is unimodal with a mode at $m = \lfloor (a-1)(b-1)/(c+1) \rfloor$. If m is integer, a second mode at $m-1$ may occur.

B. For $c > r$, the r-th factorial moment is given by $\mathbf{E}X^{(r)} = (a)_r(b)_r/(c-1)^{(r)}$, where $u^{(r)}$ denotes the r-th descending factorial.

C. The mean μ is given by $ab/(c-1)$, if $c > 1$. The variance is $\mu(1+(\mu+a+b+1)/(c-2))$ if $c > 2$. Note: I think this is wrong. It should be $\mu(1+ab/(c-2)+(a+b+1)(c-1)/(c-2))$. This is a good exercise, by the way.

D. The distribution is log-convex and J-shaped if

$$\frac{(a-1)(b-1)}{a+b+c-1} < 0 \quad \text{and} \quad \frac{(a+c)(b+c)}{a+b+c-1} > 0 \; .$$

E. As $n \to \infty$, $p_n \sim c' n^{-(1+c)}$ for some constant c'.

Random variate generation for the GHgB3 distribution.

By NB(a,p) we denote the negative binomial distribution defined by the probabilities

$$p_n = \binom{a+n-1}{n} p^a (1-p)^n (n \ge 0) \; ,$$

where $a > 0$. Similarly, Po(λ) is the Poisson distribution with parameter λ, Gamma(a,c) is the gamma distribution with parameters a,c, Beta(a,b) is the beta distribution with parameters a,b, and BeII (b,c) is the beta distribution of the second kind with density given by

$$f(x) = B^{-1}(b,c) \frac{x^{b-1}}{(1+x)^{b+c}} \; (x > 0) \; .$$

In all these cases, λ, a, b and c are positive. We have the following distributional identities:

$$\begin{aligned}
\text{GHgB3} &\stackrel{\mathcal{L}}{=} \text{NB } (a, \tfrac{1}{1+c}) \underset{c}{\wedge} \text{BeII } (b,c) \\
&\stackrel{\mathcal{L}}{=} \text{Po } (\lambda) \underset{\lambda}{\wedge} \text{Gamma } (a,c) \underset{c}{\wedge} \text{BeII } (b,c) \\
&\stackrel{\mathcal{L}}{=} \text{Po } (\lambda) \underset{\lambda}{\wedge} \text{Gamma } (a,c) \underset{c}{\wedge} \frac{\text{Gamma}(b,1)}{\text{Gamma}(c,1)} \\
&\stackrel{\mathcal{L}}{=} \text{Po } (\lambda) \underset{\lambda}{\wedge} \frac{\text{Gamma}(a,1)\text{Gamma}(b,1)}{\text{Gamma}(c,1)} .
\end{aligned}$$

It is easy to see that random variates with the GHgB3 distribution can be obtained in uniformly bounded time using one of the distributional identities given above. For all the distributions listed, efficient algorithms are available; for a survey and development of some algorithms, see Devroye (1986). For the negative binomial distribution, one could also consult Pokhodzei (1985). Good Poisson generators are given in Schmeiser and Kachitvichyanukul (1981), Ahrens and Dieter (1982) and Stadlober (1990). Efficient gamma generators are given in Marsaglia (1977), Best (1978), Cheng and Feast (1979, 1980), Ahrens and Dieter (1982), Ahrens, Kohrt and Dieter (1983) and Le Minh (1988).

Random variate generation for the digamma distribution.

The digamma distribution (Sibuya, 1979) with parameters $a, c > 0$ is defined by the probabilities

$$p_n = \frac{(a)_n}{n(a+c)_n} \frac{1}{\psi(a+c) - \psi(c)} \ (n \geq 1) \ .$$

We will write $\mathrm{DI}(a, c)$ when no confusion is possible. We recall that ψ is the standard psi-function

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{dx}} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \ .$$

1. Sibuya's generator. Sibuya (1979, p. 384) proposed the following generator based upon rejection from a GHgB3 random variate:

```
Sibuya's rejection method from a GHgB3 variate.

repeat
  Generate X as 1 plus a GHgB3(1, a+1, c-1) random variate.
  Generate U uniformly on [0,1].
until U < 1/X.
return X.
```

PROPERTY 1. *Sibuya's algorithm is valid for all $c > 1$. The expected number of iterations (N) before halting is*

$$\mathbf{E}N = \frac{a}{(c-1)(\psi(a+c)-\psi(c))} \le \frac{a+c}{c-1}\ .$$

For fixed $c > 1$, as $a \to \infty$, we have

$$\mathbf{E}N \sim \frac{a}{(c-1)\log a} \to \infty\ .$$

PROOF. Before acceptance/rejection, X has a distribution given by the probabilities

$$q_n \stackrel{\text{def}}{=} \frac{\Gamma(c)\Gamma(a+c)}{\Gamma(a+c+1)\Gamma(c-1)} \frac{(1)_{n-1}(a+1)_{n-1}}{(a+c+1)_{n-1}(n-1)!}\ (n \ge 1)\ .$$

After acceptance, the returned X has a distribution with $p_n \stackrel{\text{def}}{=} \mathbf{P}\{X = n\}$ given by the formula

$$Rp_n = q_n/n$$

where R is a constant picked to make the p_n's sum to one. The latter statement follows from the form of the acceptance condition. It is easy to verify that we can choose the p_n's as the $\mathrm{DI}(a,c)$ probabilities and that

$$R = \frac{c-1}{a}(\psi(a+c)-\psi(c))\ .$$

This follows by noting that

$$\begin{aligned}
\frac{q_n}{n} &= \frac{c-1}{a+c}\frac{1}{n!}\frac{\Gamma(n)\Gamma(a+n)\Gamma(a+c+1)}{\Gamma(1)\Gamma(a+1)\Gamma(a+c+n)} \\
&= \frac{c-1}{n}\frac{\Gamma(a+n)\Gamma(a+c)}{\Gamma(a+1)\Gamma(a+c+n)} \\
&= \frac{c-1}{an}\frac{(a)_n}{(a+c)_n} \\
&= \frac{c-1}{a}(\psi(a+c)-\psi(c))p_n\ .
\end{aligned}$$

The expected number of iterations before halting is precisely $1/R$. The statements about $\mathbf{E}N$ follow without work from the Lemmas shown below. □

LEMMA 1. *Γ is log-convex. Thus, $\log\Gamma(c+x) - \log\Gamma(x) \ge c\psi(x)$ for $c > 0$. Also, ψ is concave and strictly increasing on $[0,\infty)$. We have*

$$0 \ge \psi(x) - \log x + \frac{1}{2x} \ge -\frac{1}{12x^2}\ .$$

Furthermore, ψ' is convex and strictly decreasing on $[0,\infty)$. For $x > 0$,

$$0 \le \psi'(x) - \frac{1}{x} \le \frac{1}{x^2} \quad \text{and} \quad \psi'(x) \ge \frac{1}{x^2}\ .$$

Finally, for $c, x > 0$, we have

$$c\psi'(x) \geq \psi(x+c) - \psi(x) \geq c\psi'(x+c) \ .$$

PROOF. For most properties, we refer to Whittaker and Watson (1927). For the inequalities involving ψ, we begin with the following equality from Whittaker and Watson (1927, p.251):

$$\psi(x) = \log(x) - \frac{1}{2x} - 2\int_0^\infty \frac{t}{(t^2+x^2)(e^{2\pi t}-1)}\, dt \ .$$

The upper bound for ψ follows without work from this. The lower bound is obtained by noting that

$$\int_0^\infty \frac{t}{(t^2+x^2)(e^{2\pi t}-1)}\, dt \leq \frac{1}{x^2}\int_0^\infty \frac{t}{(e^{2\pi t}-1)}\, dt = \frac{1}{24x^2} \ .$$

The inequalities for ψ' are obtained from an equality involving the generalized zeta function found on p.250 of Whittaker and Watson (1927):

$$\psi'(x) = \sum_{n=0}^\infty \frac{1}{(x+n)^2} \geq \frac{1}{x^2} \ .$$

Clearly,

$$\begin{aligned} \frac{1}{x} &= \int_0^\infty \frac{1}{(x+t)^2}\, dt \leq \psi'(x) \\ &\leq \frac{1}{x^2} + \int_0^\infty \frac{1}{(x+t)^2}\, dt = \frac{1}{x^2} + \frac{1}{x} \ . \end{aligned}$$

The last inequalities in the Lemma can be deduced from the concavity of ψ. $\square$

LEMMA 2.

$$\psi(a+c) - \psi(c) \geq \frac{a}{c+a} \ .$$

Furthermore,

$$\psi(a+c) - \psi(c) \geq \log\left(\frac{a+c}{c}\right) \ .$$

Finally,

$$\psi(a+c) - \psi(c) \geq \frac{a}{c(a+c)} \ .$$

PROOF. The first inequality follows rather easily from Lemma 1. The second inequality is immediate from the integral representation of ψ given in the second sentence of the proof of Lemma 1. The third inequality follows from the fact that $\psi'(x) \ge 1/x^2$ (Lemma 1), and that

$$\psi(a+c) - \psi(c) = \int_c^{a+c} \psi'(y)\,dy \ge \int_c^{a+c} y^{-2}\,dy = \frac{1}{c} - \frac{1}{a+c} \,. \square$$

Per iteration, Sibuya's method uses one GHgB3 random variate. Moreover, no costly function evaluations are necessary. Unfortunately, Sibuya's algorithm can only be used if $c > 1$. Also, even if $c > 1$, the expected time is unbounded if we let $a \uparrow \infty$ while keeping c fixed. In summary, the algorithm operates well when $a + c$ is relatively small compared to $c - 1$. We also note that $\mathbf{E}N = \mu$, the mean of the distribution. Thus, whenever Sibuya's algorithm is fast, μ is small, i.e., the distribution is concentrated near the mode and short-tailed.

2. A SIMPLE REJECTION ALGORITHM.

It is possible to modify Sibuya's algorithm very slightly to make it more universally useful.

```
Rejection from a GHgB3 variate.

repeat
  Generate X as 1 plus a GHgB3 (a,1,c) random variate.
  Generate a uniform [0,1] random variate U.
until U < (a+X-1)/(max(a,1)X)   .
return X
```

Let us justify this method first. Define $\Delta = \psi(a+c) - \psi(c)$, and let p_n be the DI(a,c) probability. Noting that before the acceptance/rejection step, X is a GHgB3 $(a,1,c)$ random variate shifted by one, we see that its law is governed by the probabilities

$$\begin{aligned}
q_n &\stackrel{\text{def}}{=} \frac{\Gamma(1+c)\Gamma(a+c)}{\Gamma(a+c+1)\Gamma(c)} \frac{(1)_{n-1}(a)_{n-1}}{(a+c+1)_{n-1}(n-1)!} \quad (n \ge 1) \\
&= \frac{c}{(a+c)} \frac{(a)_{n-1}}{(a+c+1)_{n-1}} \quad (n \ge 1) \\
&= \frac{c}{(a+c+n-1)} \frac{(a)_{n-1}}{(a+c)_{n-1}} \quad (n \ge 1) \\
&= p_n \frac{c\Delta n}{(a+c+n-1)} \frac{(a)_{n-1}(a+c)_n}{(a)_n(a+c)_{n-1}} \quad (n \ge 1) \\
&= p_n \frac{c\Delta n}{(a+n-1)} \quad (n \ge 1)
\end{aligned}$$

We see that

$$\sup_n \frac{p_n}{q_n} = \begin{cases} \frac{1}{c\Delta} & \text{when } a < 1 \text{ (reached as } n \to \infty) \\ \frac{a}{c\Delta} & \text{when } a \geq 1 \text{ (reached as } n = 1) \end{cases}$$

$$= \frac{\max(a,1)}{c\Delta} \ .$$

Thus, the random variate X in the algorithm should be accepted with probability $(a + X - 1)/(X \max(a,1))$.

PROPERTY 2. *The expected number of iterations* ($\mathbf{E}N$) *is given by*

$$\mathbf{E}N = \frac{\max(a,1)}{c\Delta} = \frac{\max(a,1)}{c(\psi(a+c) - \psi(c))} \ .$$

Also,

$$\mathbf{E}N \leq \max(1,a) \min(1,1/c) + max(1,1/a) \min(1,c) \ .$$

PROOF. By Lemma 2,

$$\frac{1}{c\Delta} \leq \frac{c+a}{a \max(1,c)} = (1/a) \min(1,c) + \min(1,1/c) \ . \ \square$$

REMARK 1. Since $\min(a,1) \leq c\Delta \leq \max(a,1)$, the rejection constant does not exceed $\max(a,1/a)$. For fixed a, the algorithm is uniformly fast in c. $\square$

REMARK 2. For $a = 1$, there is no rejection. This implies that

$$\mathrm{DI}(1,c) \stackrel{\mathcal{L}}{=} 1 + \mathrm{GHgB3}(1,1,c) \stackrel{\mathcal{L}}{=} 1 + \mathrm{Po}(\lambda) \bigwedge_{\lambda} EE'/G_c \ ,$$

where E, E' are independent exponential random variables, and G_c is a gamma(c) random variable. In other words, we recover the fact that a digamma($1,c$) random variable is distributed as one plus a Yule(c) random variable Y, where

$$\mathbf{P}\{Y = n\} = \frac{c\Gamma(c+1)n!}{\Gamma(n+c+2)} = \frac{c\ n!}{(1+c)_{n+1}} \ (n \geq 0) \ . \ \square$$

3. REJECTION WITH TWO DOMINATING CURVES: CASE $\mathbf{a \geq c+1}$. We will use a slightly more sophisticated dominating curve, taking into account the following inequalities:

$$p_n \leq \begin{cases} \frac{1}{n\Delta} \\ q_n \times \frac{a+n^*}{(1+n^*)c\Delta} & (n > n^*) \end{cases}$$

where n^* is an integer to be picked further on, and q_n is the probability that one plus a GHgB3$(a,1,c)$ random variate is n. Note that the second bound requires $a \geq 1$. To simplify the design of the rejection algorithm, we will replace the first upper bound by one that is about $\log 2$ looser. Define the following quantities:

$$t = (a-1)/c \; ;$$
$$u = \lfloor \log_2 t \rfloor \; ;$$
$$n^* = 2^{u+1} - 1 \; .$$

The condition $a \geq c+1$ assures that $u \geq 0$ and $t \geq 1$. For $1 \leq n \leq n^*$, we employ the inequality

$$p_n \leq \frac{1}{\Delta 2^{\lfloor \log_2 n \rfloor}} \; .$$

This has two advantages: first, the sum of the values of the upper bound taken between 1 and n^* is simply $(u+1)/\Delta$. Secondly, a random variate with probability vector proportional to the upper bound can be generated by an alias method trick shown on pages 111-112 of Devroye (1986). For $u \geq 0$, proceed as follows:

```
Generator for the semi-harmonic distribution on {1,...,n*}.

Generate Y uniformly on {0,1,2,...,u}.
Generate X uniformly on {2^Y,...,2^(Y+1) - 1}.
return X
```

The joint algorithm is as follows:

```
[Set-up]
t = (a − 1)/c
u = ⌊log_2 t⌋
n* = 2^{u+1} − 1
Δ ← ψ(a + c) − ψ(c)
w ← (u+1)/Δ, z ← (a + n*)/(cΔ(n* + 1))

[Generator]
repeat
  Generate U uniformly on [0,1].
  if U < w/(w + z)
    then Generate X with the semi-harmonic distribution on {1,...,n*}.
         (Generate Y uniformly on {0,1,2,...,u}.)
         (Generate X uniformly on {2^Y,...,2^{Y+1} − 1}.)
         Generate a uniform [0,1] random variate V.
         W ← V/(Δ2^Y) .
         Accept ← [W < p_X].
         (Equivalently, Accept ← [V < 2^Y (a)_X/(X(a + c)_X)].  )
    else Generate Y with the GHgB3(a,1,c) distribution
         Set X ← Y + 1.
         Generate a uniform [0,1] random variate V.
         Accept ← [X > n*] ∩ [Vz < (a + X − 1)/(cΔX)].
until Accept
return X
```

PROPERTY 3. *If* $a \geq c+1$ *and* $a/c \geq e-1$, *then*

$$\mathbf{E}N \leq \log_2 e + 3 \ ,$$

where $\mathbf{E}N$ *is the expected number of iterations.*

PROOF. The rejection constant $\mathbf{E}N$ is

$$\begin{aligned}\frac{u+1}{\Delta}+\frac{a+n^*}{c\Delta(n^*+1)} &= \frac{\lfloor\log_2\left(\frac{a-1}{c}\right)\rfloor+1}{\Delta}+\frac{a+2^{u+1}-1}{c\Delta 2^{u+1}} \\ &\le \frac{\log_2\left(\frac{a-1}{c}\right)+1}{\Delta}+\frac{1}{c\Delta}+\frac{a-1}{c\Delta 2^{u+1}} \\ &\le \frac{\log_2(a-1)-\log_2 c+1}{\log(a+c)-\log c}+\frac{1}{c\Delta}+\frac{1}{\Delta} \\ &\le \log_2 e+1+\frac{1}{c\Delta}+\frac{1}{\Delta}\ .\end{aligned}$$

We recall that in general $c\Delta \ge \min(a,1)$. Also, from Lemma 2, $\Delta \ge \log((a+c)/c) \ge 1$ if $a/c \ge e-1$. Thus,

$$\mathbf{E}N \le \log_2 e+3\ .\ \square$$

We conclude that the expected number of iterations of the above algorithm is uniformly bounded over all (a,c) with $a \ge c+1$ and $a/c \ge e-1$. The disadvantage is that at one point we need to evaluate gamma functions (see $[W < p_X]$).

4. A special method for small values of a.

We begin with the following inequality:

LEMMA 3.

$$p_n \le \frac{D}{n^{1+c}}\ (n \ge 1)$$

where

$$D \stackrel{\text{def}}{=} \max\left(\frac{a}{\Delta(a+c)}\,,\ \frac{\Gamma(a+c)e^{c/(2(a+1))}e^{c/(12(a+1)^2)}}{\Gamma(a)\Delta}\right)$$

and $\Delta=\psi(a+c)-\psi(c)$.

PROOF. For $n=1$, we note that

$$p_1 = \frac{a}{\Delta(a+c)} \ ,$$

so that the inequality is verified. For $n \geq 2$, we note the following chain of inequalities: Applying Lemma 1, we have

$$\begin{aligned}
p_n &= \frac{(a)_n}{n(a+c)_n\Delta} \\
&= \frac{\Gamma(a+n)\Gamma(a+c)}{\Gamma(a)\Gamma(a+c+n)n\Delta} \\
&= \frac{\Gamma(a+c)}{\Gamma(a)n\Delta} \exp\left(-\int_{a+n}^{a+c+n} \psi(y)\,dy\right) \\
&\leq \frac{\Gamma(a+c)}{\Gamma(a)n\Delta} \exp\left(-c\psi(a+n)\right) \\
&\qquad \text{(This uses the fact that } a+n \geq 2.\text{)} \\
&\leq \frac{\Gamma(a+c)}{\Gamma(a)n\Delta}(a+n)^{-c} e^{c/2(a+n)} e^{c/(12(a+n)^2)} \\
&\leq \frac{D}{n^{1+c}} \ . \ \square
\end{aligned}$$

The bound of Lemma 3 suggests the use of the rejection method with dominating density

$$f(x) = \frac{c}{x^{1+c}} \ (x \geq 1) \ ,$$

which has distribution function $F(x) = 1 - 1/x^c$. To obtain the proper rejection constant, we require

$$\sup_{n\geq 1} \frac{p_n}{f(n+1)} = \sup_{n\geq 1} \frac{D}{c}\left(\frac{n+1}{n}\right)^{1+c} = A \stackrel{\text{def}}{=} \frac{2^{1+c}D}{c} \ .$$

Weaponed with the inequality

$$p_n \leq A \inf_{n\leq x<n+1} f(x) \ ,$$

we can apply the following rejection algorithm.

```
repeat
  Generate U,V uniformly on [0,1].
  Y ← 1/(1 − U)^{1/c}
  X ← ⌊Y⌋
  W ← V A f(Y) = V(2/Y)^{1+c} D
until W < p_X
return X
```

PROPERTY 4. *For the algorithm shown above,*

$$\mathbf{E}N = 2^{1+c}D/c \, .$$

$\mathbf{E}N$ *is uniformly bounded over all* $a \in (0, a_0]$ *and* $c \in (0, c_0]$ *where* a_0 *and* c_0 *are finite constants.*

PROOF. Note that 2^{1+c} is bounded over the given domain. So, we need only establish the uniform boundedness of

$$I \stackrel{\text{def}}{=} \frac{a}{\Delta c(a+c)}$$

and

$$II \stackrel{\text{def}}{=} \frac{\Gamma(a+c)e^{c/(2(a+1))}e^{c/(12(a+1)^2)}}{\Gamma(a)c\Delta} \, .$$

But $I \leq 1$ by Lemma 2. The two exponents in II are uniformly bounded. Using the same inequality for Δ once more, we see that

$$\frac{\Gamma(a+c)}{\Gamma(a)c\Delta} \leq \frac{(a+c)\Gamma(a+c)}{a\Gamma(a)} = \frac{\Gamma(a+c+1)}{\Gamma(a+1)} \, ,$$

which is bounded in a uniform manner over the given collection of parameters. □

5. OTHER METHODS. Sibuya (1979, Theorem 4) pointed out that a digamma (a, c) random variate is distributed like a logarithmic series variate with parameter $1 - Y$, where Y is end-accented-beta, a distribution with a density related to the beta density. The generation of a random variate with the latter distribution seems to be about as hard as the direct generation of digamma variates, so this detour does not seem to be promising for good speed. Noting that p_n is $\downarrow$, one can also employ a general algorithm for monotonically decreasing densities when the first moment μ is known:

$$\mu = \frac{a}{(c-1)\Delta} \, .$$

The general rejection method of Devroye (1986, p.313) when applied to discrete distributions leads to the inequality

$$p_n \leq \frac{2\mu}{n(n+1)} \ (n \geq 1) \ .$$

If this bound is used in a rejection algorithm, the expected number of iterations is given by

$$\mathbf{E}N = 2\mu \leq 2\frac{a+c}{c-1} \ .$$

Not unexpectedly, this algorithm is comparable to and worse than the algorithm of section 1 that was specifically designed for the digamma distribution. For the sake of completeness, here is the algorithm:

> `repeat`
>   `Generate uniform` $[0,1]$ `random variates` U, V.
>   `Set` $X \leftarrow \lfloor 1/U \rfloor$.
> `until` $2\mu V/(X(X+1)) \leq p_X$.
> `return` X

Finally, the inversion of a uniform random variate based upon on-line computations of

$$\frac{p_{n+1}}{p_n} = \frac{a+n}{a+n+c}\frac{n}{n+1}$$

leads to a time complexity that grows as μ as well. Again, this is of the same order of magnitude as the previous algorithm.

```
Generate a uniform [0,1] random variate U.
P ← p_1 = a/((a+c)Δ)  (P denotes ``probability''.)
S ← P.  (S stands for ``sum''.)
X ← 1.
while U > S do
   P ← PX(a+X)/((X+1)(a+X+c)).
   S ← S+P.
   X ← X+1.
return X
```

6. Comparison of the algorithms. In figure 1, we show the areas in the parameter space in which the algorithms of this paper are dominant. Taken together, the different properties show that $\min(\mathbf{E}N_1, \mathbf{E}N_2, \mathbf{E}N_3, \mathbf{E}N_4)$ remains uniformly bounded over all values of a, c, where N_i denotes the expected number of iterations of the rejection algorithm of section i. More often than not, users will be interested in particular areas of the parameter space, so that one or two methods suffice to obtain good expected time. Figure 1 was obtained by computing all the values $\mathbf{E}N_i$ at points of the grid defined by $a = 2^k$, $c = 2^l$, with k and l integers drawn from $\{-5, \ldots, 5\}$. Between grid points, we just drew rough polygonal approximations. Interestingly, at the 121 grid points, the best value of $\mathbf{E}N_i$ was less than 2 except for those points in the region dominated by the method of section 4, where the values were between 2 and 4.42. Over that grid, the maximum for the minimal $\mathbf{E}N_i$ was 4.413269612, observed at $a = 1/8$, $c = 1$. Note also that in some cases, one can establish the dominance of one method over another theoretically. For example, method 1 is dominated by method 2 when $a > 1 - 1/c$, which explains the shape of the boundary between these methods.

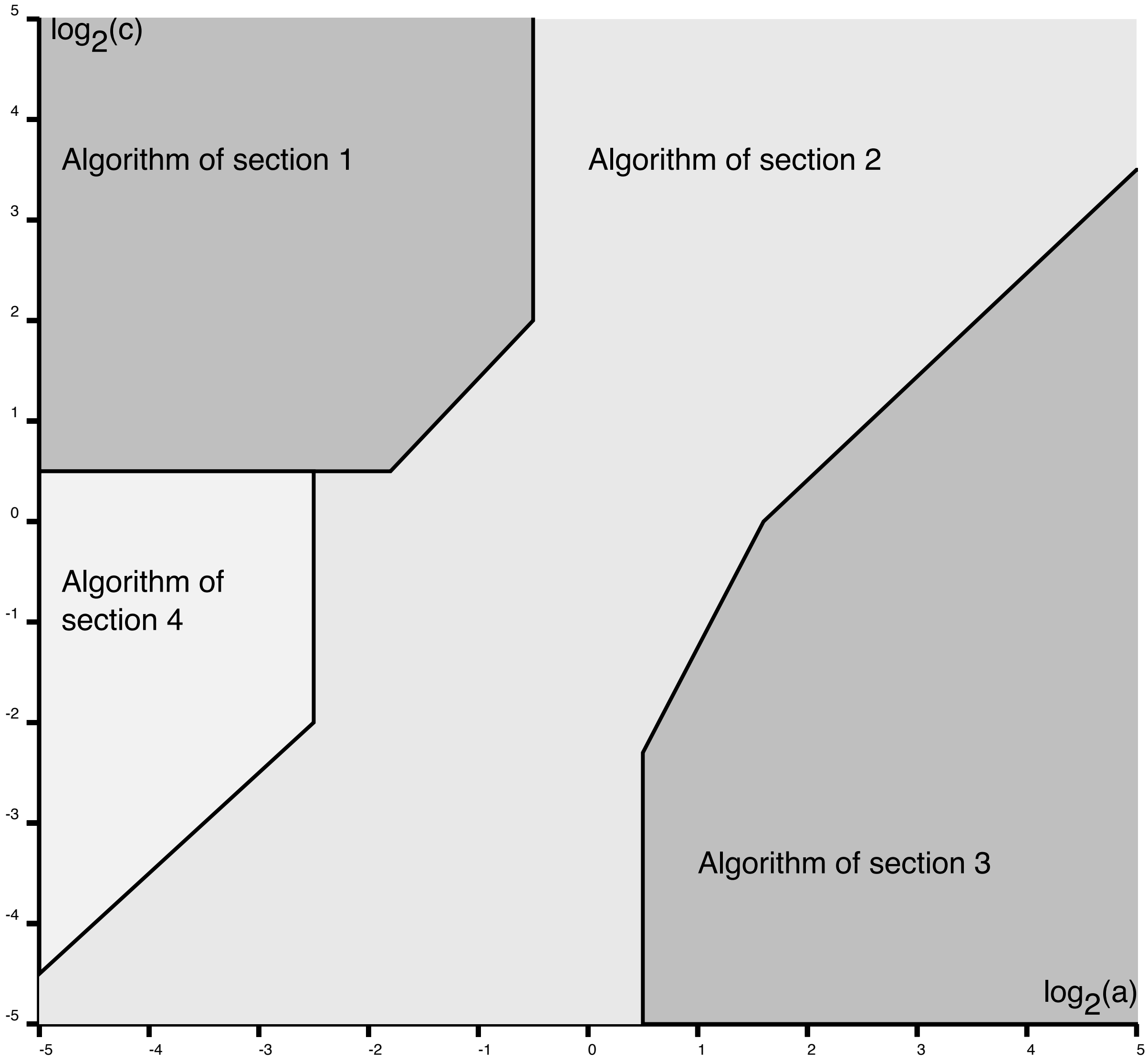

Figure 1. Regions of dominance of algorithms are shown.

Figure 1.

Random variate generation for Sibuya's trigamma distribution.

Sibuya's trigamma distribution with parameter $c > 0$ is obtained from the digamma distribution by letting $a \downarrow 0$. The probabilities are given by

$$p_n = \frac{(n-1)!}{\psi'(c)n(c)_n} \ (n \geq 1).$$

The mean is $\mu = 1/(\psi'(c)(c-1))$. In the plane of the parameters of the digamma distribution, the trigamma occupies the x-axis. Thus, two algorithms immediately impose themselves. We will give the algorithms and their analyses without providing the rather trivial details. The first algorithm is more efficient than thex second one for c greater than about 1.14.

```
Trigamma generator for large values of c

repeat
  Generate X as 1 plus a GHgB3(1, 1, c − 1) random variate.
  (Note:  X is distributed as Po(λ) ∧_λ EE'/G_{c−1}
  where E, E' are exponential random variables,
  and G_{c−1} is a gamma (c − 1) random variable.)
  Generate U uniformly on [0, 1].
until U < 1/X.
return X.
```

Property 5. *The algorithm is valid for $c > 1$. The expected number of iterations (N) before halting is*

$$\mathbf{E}N = \frac{1}{(c-1)\psi'(c)} \leq \frac{c}{c-1} \ .$$

For the second algorithm, we require a simple inequality:

Lemma 4.

$$p_n \leq \frac{D}{n^{1+c}} \ (n \geq 1)$$

where

$$D \stackrel{\text{def}}{=} \max\left(\frac{1}{c\psi'(c)}, \frac{\Gamma(c)e^{c/2}e^{c/12}}{\psi'(c)}\right) \ .$$

```
Trigamma generator for small values of c

repeat
  Generate U,V uniformly on [0,1].
  Y ← 1/(1 − U)^{1/c}
  X ← ⌊Y⌋
  W ← V(2/Y)^{1+c} D
until W < p_X
return X
```

PROPERTY 6. *For the algorithm shown above,*

$$\mathbf{E}N = 2^{1+c}D/c \ .$$

$\mathbf{E}N$ *is uniformly bounded over all* $c \in (0, c_0]$ *where* c_0 *is a finite constant.*

PROOF. Note that 2^{1+c} is bounded over the given domain. So, we need only establish the uniform boundedness of

$$I \stackrel{\text{def}}{=} \frac{1}{c^2\psi'(c)}$$

and

$$II \stackrel{\text{def}}{=} \frac{\Gamma(c)e^{c/2}e^{c/12}}{c\psi'(c)} \ .$$

We have $\psi'(c) \geq 1/c^2$. Thus, $I \leq 1$. Also,

$$II \leq c\Gamma(c)e^{c/2}e^{c/12} = \Gamma(c+1)e^{7c/12} \ . \square$$

Relationship with the logarithmic series distribution.

Sibuya (1979, Theorem 4) pointed out that a digamma (a, c) random variable is distributed as a logarithmic series random variate with parameter $1 - Y$, where Y itself is EABe (c, a), the EABe (end-accented beta) distribution has density

$$f(x) = g_{c,a}(x) \frac{\log(\frac{1}{x})}{\psi(c+a) - \psi(c)}, 0 \le x \le 1,$$

and $g_{c,a}$ is the beta density with parameters c and a. Setting $a = 0$ in this recipe yields a trigamma random variate. The EABe $(c, 0)$ density is

$$f(x) = \frac{\log(\frac{1}{x}) x^{c-1}}{\psi'(c)(1-x)}, 0 \le x \le 1.$$

If X has EABe $(c, 0)$ density, then $Z = \log(1/X)$ has density

$$h(z) = z e^{-cz} / (\psi'(c)(1 - e^{-z})) \quad (z > 0) \ .$$

We have

$$h(z) \le \begin{cases} (2/\psi'(c)) e^{-cz} & (z \le 1) \\ (e/((e-1)c\psi'(c))) cz e^{-cz} & (z > 1) \end{cases}$$

If we use rejection from the suggested truncated exponential on $[0, 1]$, and from the gamma (2) dominating curve elsewhere, then the rejection constant is

$$\frac{2(1 - e^{-c})}{c\psi'(c)} + \frac{e}{(e-1)c^2 \psi'(c)} \le 2(1 - e^{-c}) + \frac{e}{e-1} \le 2 + \frac{e}{e-1}.$$

Here we recall that $\psi'(c) \ge \max(1/c, 1/c^2)$. Thus, random variates with density h can be obtained in uniform time for all $c > 0$. Random variate generation for the logarithmic series distribution in time uniformly bounded over all parameters is discussed in section X.5.1 of Devroye (1986).

The discrete stable distribution.

Steutel and Van Harn (1979) introduced the discrete stable family via the generating function

$$g(s) = \exp(-\lambda(1-s)^\gamma), \ |s| \le 1 \ ,$$

where $\lambda > 0$ and $\gamma \in (0, 1]$ are two parameters. For $\gamma = 1$, we obtain the Poisson distribution with parameter λ. Hansen (1988) pointed out that p_n is log-convex if and only if $\lambda\gamma < 1 - \gamma$; in those cases, we thus have a long-tailed distribution. Note also that the discrete stable distribution is a compound Poisson distribution, so that random variates can be obtained as follows:

$$\sum_{1 \le i \le Y} Z_i,$$

where Y is Poisson (λ), and $Z_1, Z_2, \ldots$ are i.i.d. discrete random variables with generating function

$$h(s) = 1 - (1-s)^\gamma .$$

This is best seen by noting that

$$g(s) = \mathbf{E}(s^X) = \mathbf{E}\left(\prod_{i \le Y} s^{Z_i}\right) = \mathbf{E}(\mathbf{E}^Y(s^{Z_1})) = \mathbf{E}(h^Y(s)) = e^{\lambda(h(s)-1)} = e^{-\lambda(1-s)^\gamma} .$$

If we can generate the Z_i's at unit expected time cost, then the expected time for this method is the expected time needed to generate Y plus λ. This is $O(1+\lambda)$, since all known algorithms for the Poisson distribution take expected time bounded by $O(1+\lambda)$. From the binomial expansion, we see immediately that the probabilities corresponding to the generating function $h(s)$ are given by

$$q_n = \frac{\gamma(1-\gamma)\cdots(n-1-\gamma)}{n!} = \frac{\gamma(1-\gamma)_{n-1}}{n!} \ (n > 1) \ .$$

We have $q_0 = 0$ and $q_1 = \gamma$. For $\gamma = 1$, the distribution is monoatomic with $q_1 = 1$. Interestingly, we can generate Z_i as one plus a GHgB3 $(1, 1-\gamma, \gamma)$ random variable (Sibuya, 1979). Thus, the Z_i's are indeed available at a cost uniformly bounded over γ. Sibuya (1979) also pointed out that the distribution determined by $\{q_n\}$ is the limit of a digamma (a, γ) distribution as $a \to -\gamma$. This is a case not considered above.

It is also possible to generate the Z_i's directly by the rejection method. We note that

$$\frac{q_{n+1}}{q_n} = \frac{n-\gamma}{n+1} < 1.$$

Thus, $q_n \downarrow 0$. Furthermore, the distribution is log-convex as it is easy to check that $q_{n+1}^2 \le q_{n+2} q_n$ for all $n \ge 1$. This implies that the distribution is long-tailed, so a careful treatment is necessary. A dominating curve can be obtained as follows:

$$q_n = \frac{\gamma}{n} \prod_{i=1}^{n-1} (1 - \frac{\gamma}{i}) \le \frac{\gamma}{n} e^{-\gamma H_{n-1}} \le \frac{\gamma}{n} e^{-\gamma \log n} = \frac{\gamma}{n^{1+\gamma}} .$$

Here H_n is the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. This bound suggests a rejection algorithm from a continuous distribution with density f proportional to $\min(\gamma, \gamma / x^{1+\gamma})$ on $[0, \infty)$. To generate a random variate with density f, consult Devroye (1986). We thus have

```
repeat
  Generate X with density f, set Y ← ⌈X⌉.
  Generate a uniform [0,1] random variate U.
  Set T ← U min(γ, γ/X^{1+γ}).
  IF T > γ/Y^{1+γ}
    then Accept ← False
    else Accept ← [T < q_Y]
until Accept
return Y.
```

In the algorithm shown above, the expected number of iterations is precisely $1+\gamma$. The only technical point not covered yet is that of the evaluation of q_n. To this end, we observe that

$$q_n = \frac{\gamma\Gamma(n-\gamma)}{\Gamma(1-\gamma)\Gamma(n+1)}.$$

The evaluation of the logarithm of this can be carried out in such a manner that the expected time per evaluation in our algorithm is $O(1)$ (see Devroye, 1986, chapter X). Some care should be taken here when Stirling's approximation is used, since n is usually small in our case, and Stirling's approximation is not very accurate for small n. This can be circumvented by using Binet's convergent series. To speed up matters, we can put in a quick acceptance step, and for the perfectionist, the acceptance step can be replaced by a series of acceptance and rejection steps as in the series method, with the bounds being derived from Binet's convergent series for the gamma function.

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