EXTREMAL PARAMETERS IN CRITICAL AND SUBCRITICAL GRAPH CLASSES

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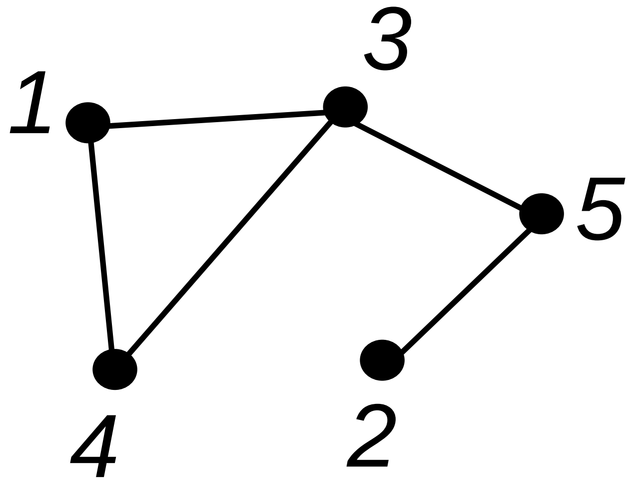
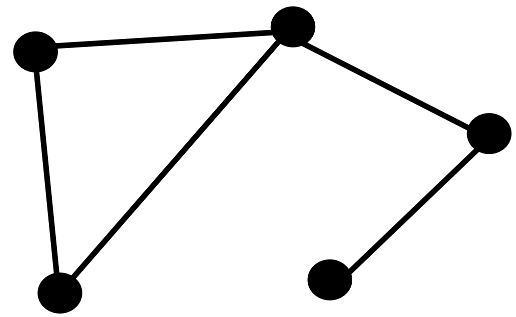
Summary

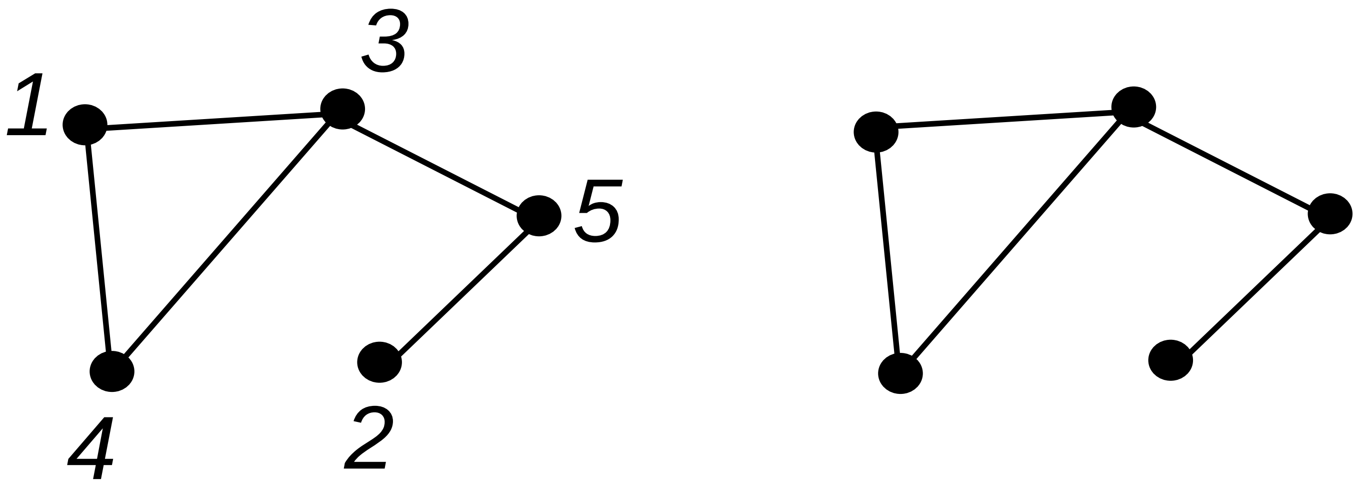
- Block-Decompostion of Graphs
- Critial and Subcritical Graph Classes
- Additive Parameters in Subcritical Graph Classes
- **Extremal Parameters in Subcritical Graph Classes**
- Additive and Extremal Parameters in Planar Maps
- **The Maximum Degree in Random Planar Graphs**

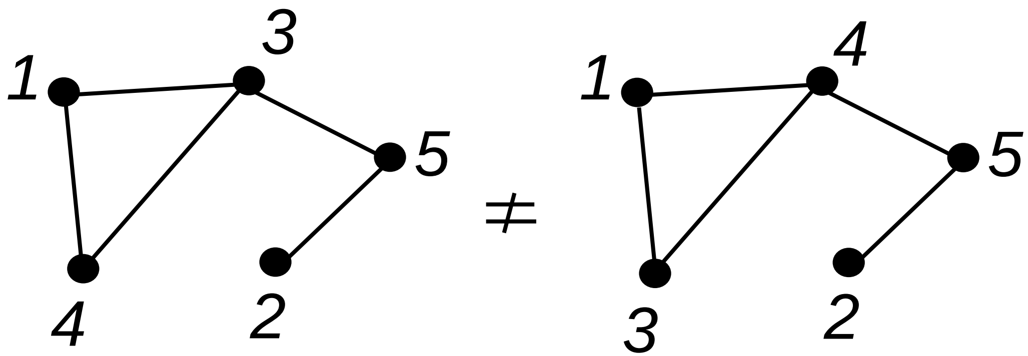
Labelled vs. Unlabelled Graphs



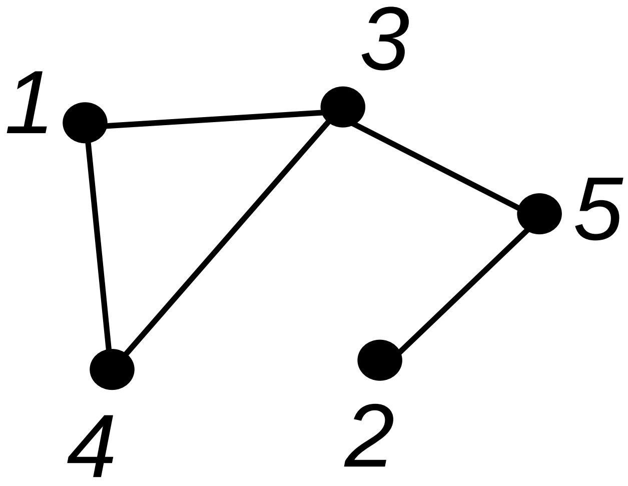


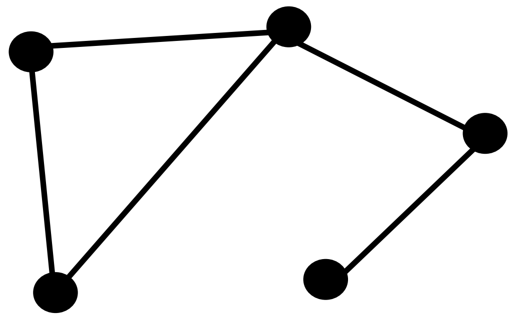
Labelled vs. Unlabelled Graphs





Labelled vs. Unlabelled Graphs





$$\frac{x^5}{5!}$$

$$x^5$$

Generating Functions

g_n ... number of graphs of size n (in a given graph class)

Labelled Graphs

$$\boxed{G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}}$$

Unlabelled Graphs

$$\boxed{G(x) = \sum_{n \geq 0} g_n x^n}$$

Generating Functions – Extensions

$g_{n,m}$ ... number of graphs of size n with m edges

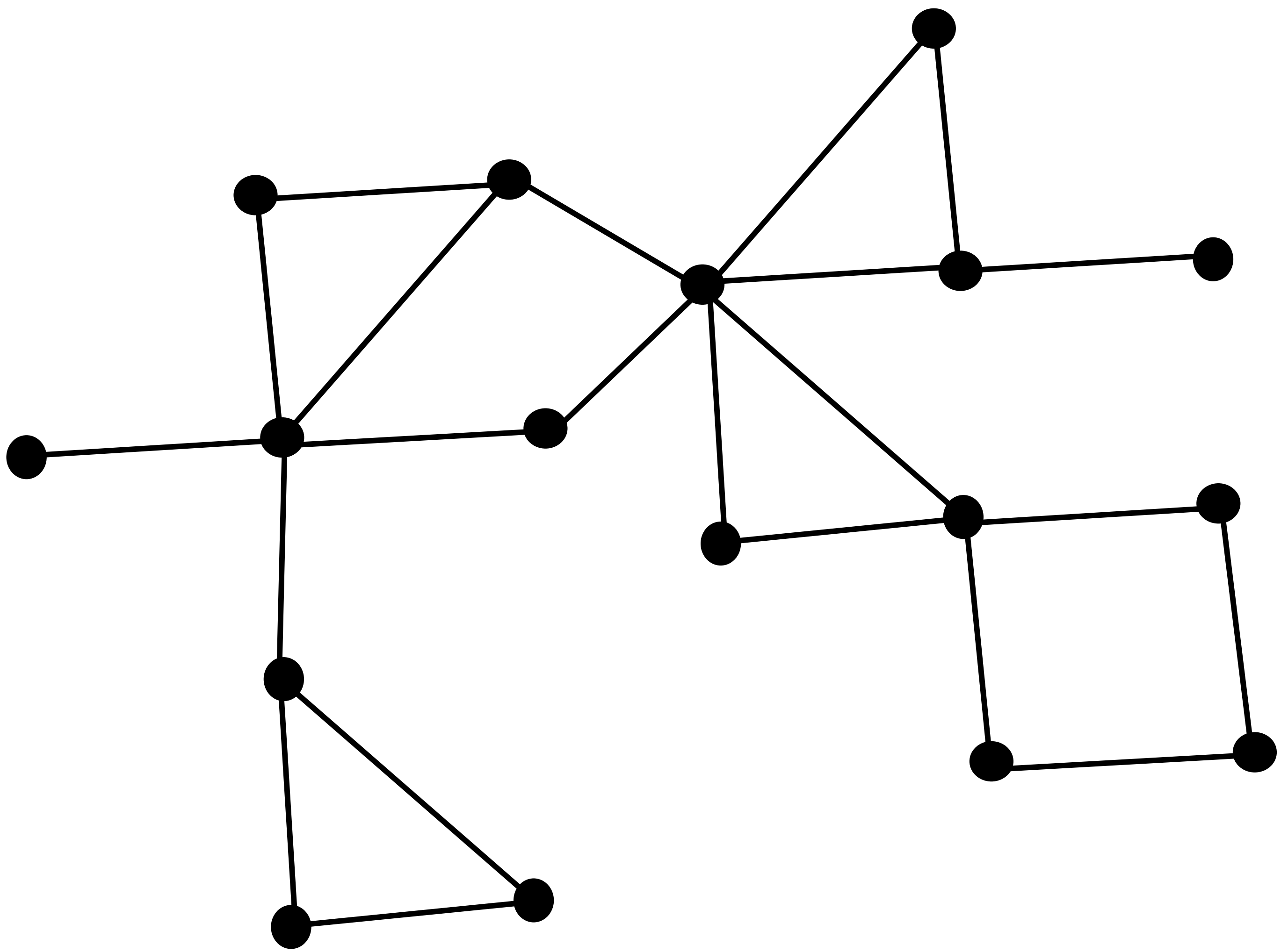
Vertext-labelled Graphs with unlabelled edges

$$G(x,y) = \sum_{n,m \geq 0} g_{n,m} \frac{x^n}{n!} y^n$$

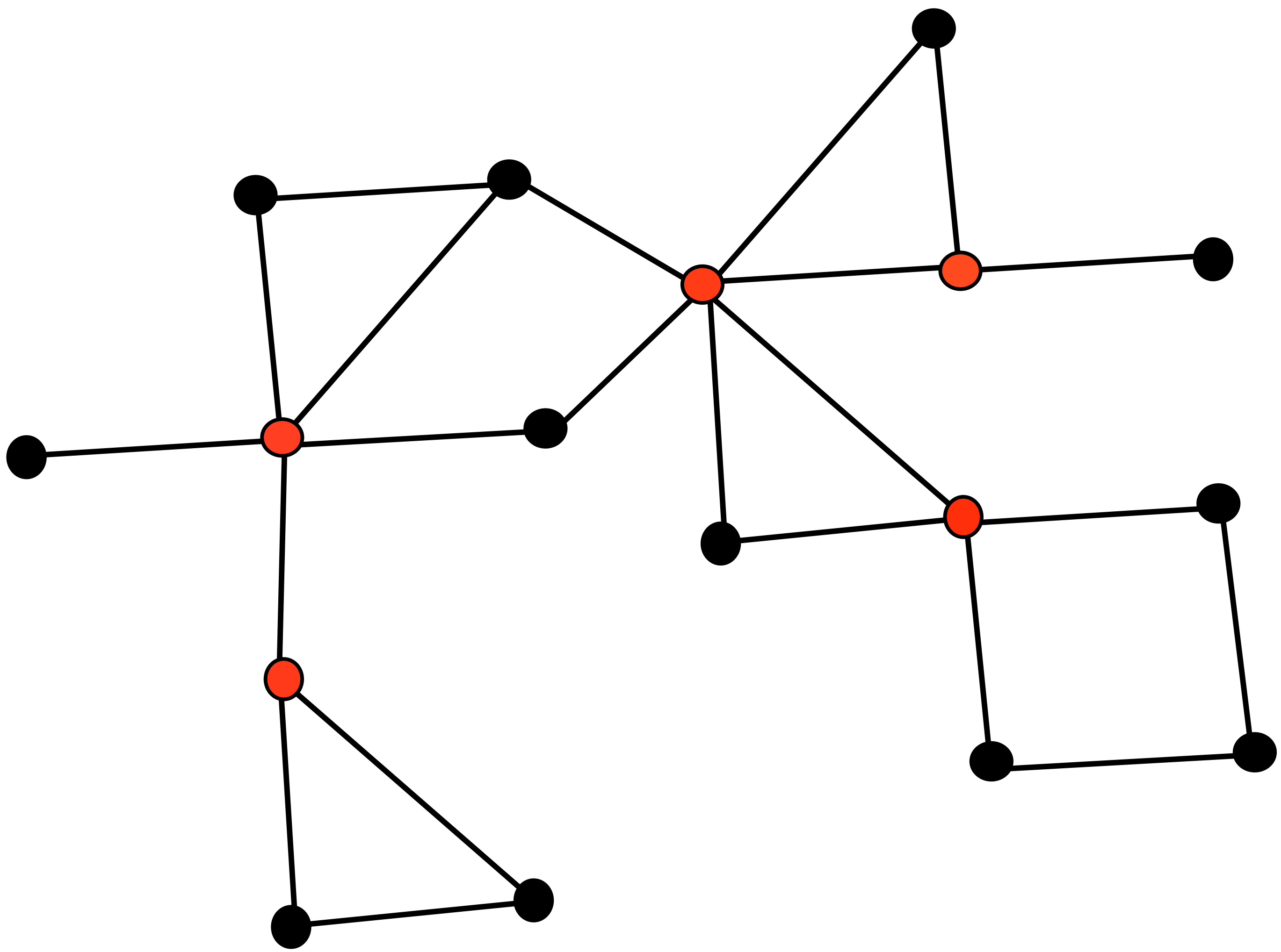
Unlabelled Graphs

$$G(x,y) = \sum_{n,m \geq 0} g_{n,m} x^n y^m$$

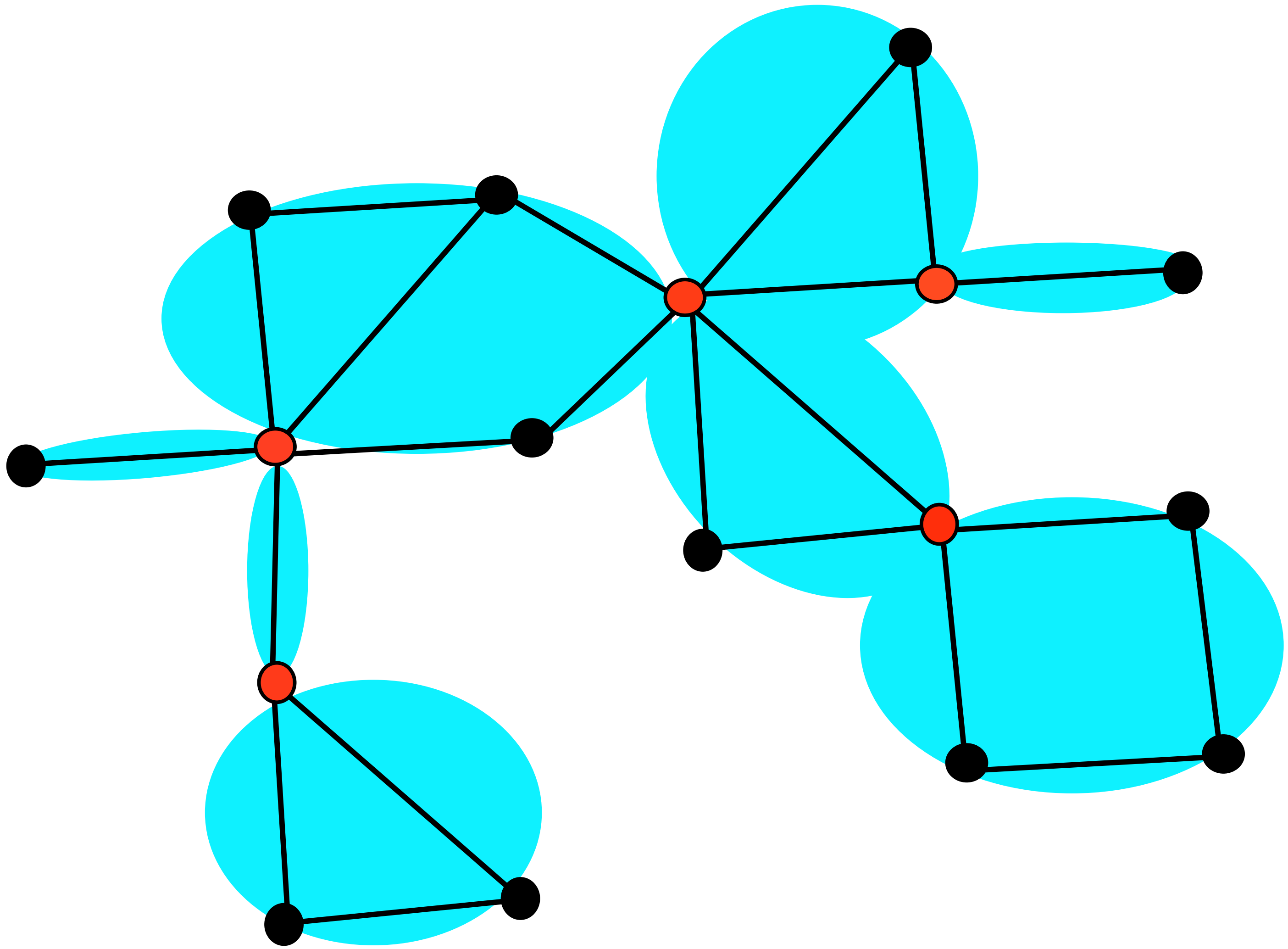
Block-Decomposition



Block-Decomposition



Block-Decomposition



Block-Decomposition

block: 2-connected component

Block-stable graph class $\mathcal{G}$: all components and all 2-connected components of a graph $G \in \mathcal{G}$ are also contained in $\mathcal{G}$

Examples: Planar graphs, series-parallel graphs, minor-closed graph classes etc.

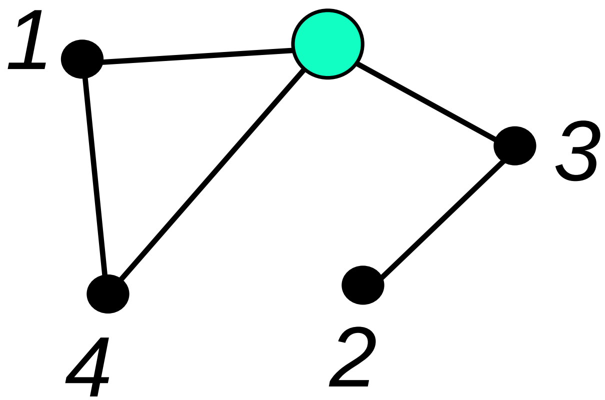
$B(x)$ ... GF for 2-connected graphs in $\mathcal{G}$

$C(x)$ ... GF for connected graphs in $\mathcal{G}$

$G(x)$ ... GF for all graphs in $\mathcal{G}$

Generating Functions for Block-Decomposition

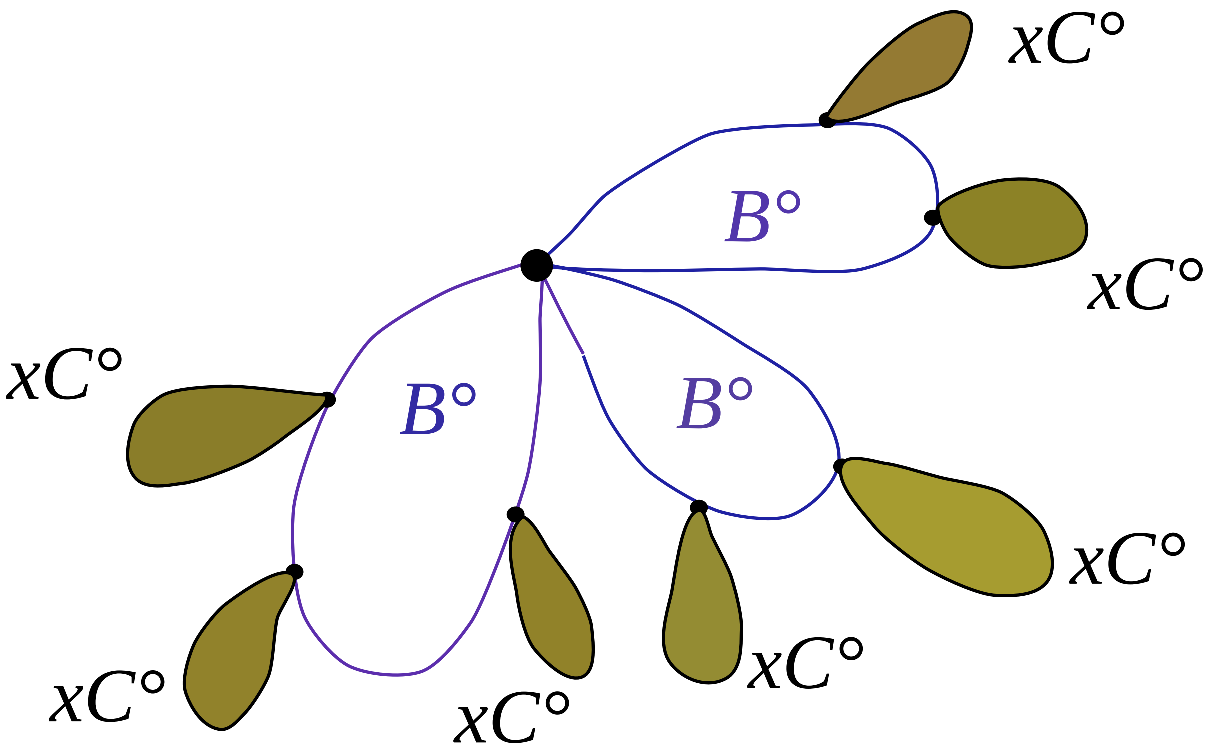
Vertex-rooted graphs: one vertext (the **root**) is distinguished (and usually discounted, that is, it gets no label)



Generating function: (in den **labelled** case)

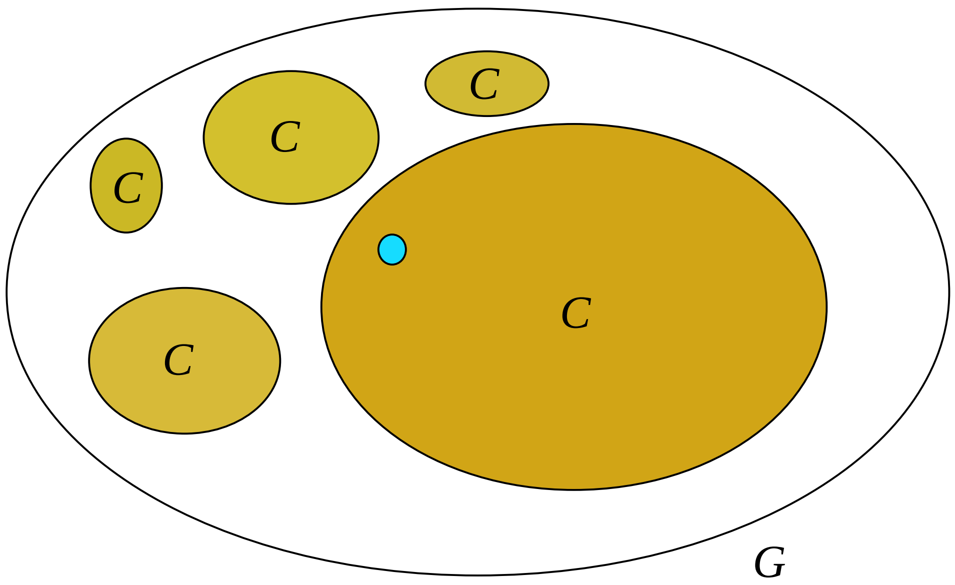
$$\boxed{G^\bullet(x) = G'(x)}$$

Generating Functions for Block-Decomposition



$$C^{\bullet}(x) = e^{B^{\bullet}(xC^{\bullet}(x))}$$

Generating Functions for Block-Decomposition



$$\boxed{G^{\bullet}(x) = \exp\left(C(x)\right)\ C^{\bullet}(x)} \iff \boxed{G(x) = e^{C(x)}}$$

Labelled Trees

Rooted Trees:

$$B^{\bullet}(x) = x$$

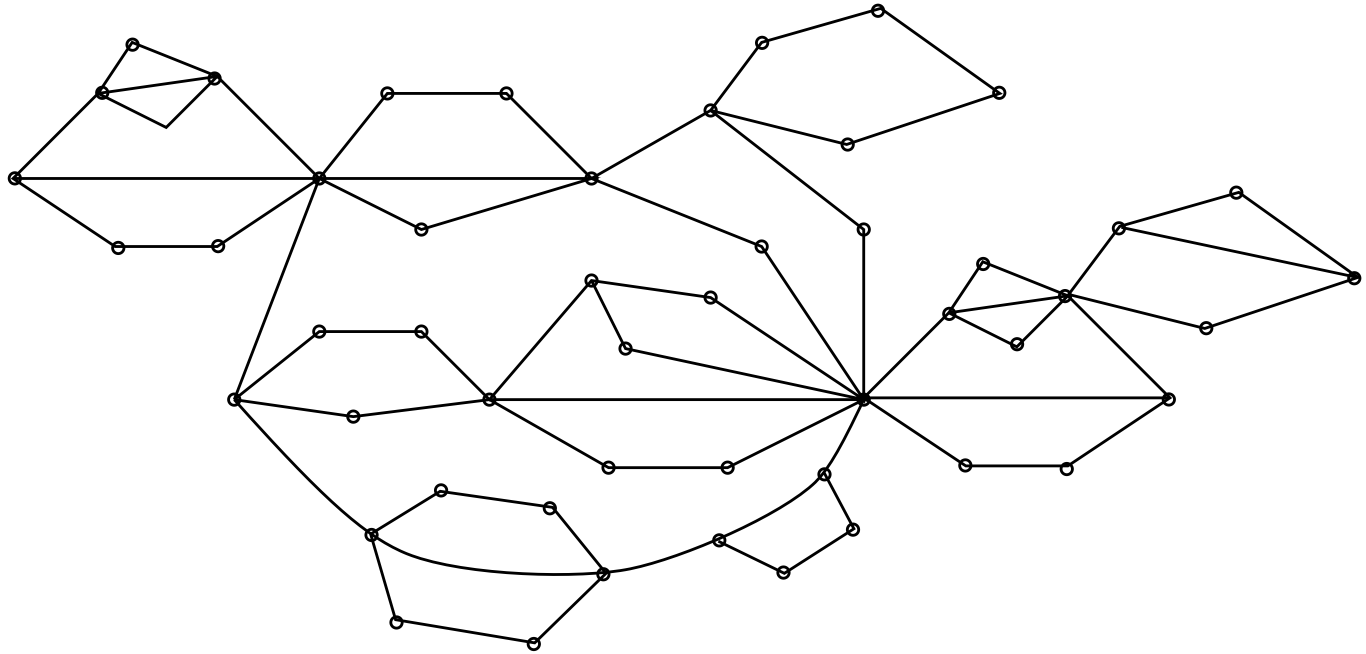
$T(x) = xC^{\bullet}(x)$ ... generating function of **rooted, labelled trees**

$$C^{\bullet}(x) = e^{B^{\bullet}(xC^{\bullet}(x))} \Longrightarrow \boxed{T(x) = xe^{T(x)}}$$

Remark: $\tilde{T}(x)$ ... GF for unrooted labelled trees:

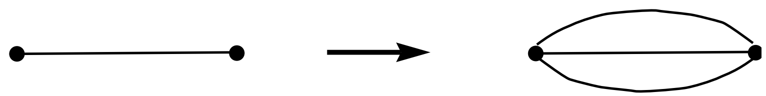
$$\tilde{T}(x)' = \frac{1}{x}T(x) \quad \Longrightarrow \quad \tilde{T}(x) = T(x) - \frac{1}{2}T(x)^2$$

Series-Parallel Graphs



Series-parallel extension of a tree or forest

Series-extension: 

Parallel-extension: 

Series-Parallel Graphs

Generating functions

$b_{n,m}$ ... number of **2-connected labelled series-parallel** graphs with n vertices and m edges

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m$$

$c_{n,m}$ ... number of **connected labelled series-parallel** graphs with n vertices and m edges

$$C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m$$

$g_{n,m}$ ... number of **labelled series-parallel** graphs with n vertices and m edges

$$G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m$$

Series-Parallel Graphs

Generating functions

$$G(x,y) = e^{C(x,y)}$$

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

$$D(x,y) = (1+y)e^{S(x,y)} - 1,$$

$$S(x,y) = (D(x,y) - S(x,y))xD(x,y).$$

Labelled Planar Graphs

$$
\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right), \\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right), \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y}, \\
\frac{M(x,D)}{2x^2 D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}, \\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right), \\
U &= xy(1+V)^2, \\
V &= y(1+U)^2.
\end{aligned}
$$

Critical vs. Subcritical Graphs

Functional equations

Suppose that $\boxed{A(x) = \Phi(x, A(x))}$, where $\Phi(x, a)$ has a power series expansion at $(0, 0)$ with non-negative coefficients and $\Phi_{aa}(x, a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (**inside the region of convergence of Φ**) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, a_0), \quad 1 = \Phi_a(x_0, a_0)}.$$

Then there exists analytic function $g(x), h(x)$ such that locally

$$\boxed{A(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}}}.$$

Remark. If there is no x_0, a_0 inside the region of convergence of Φ then the singular behaviour of Φ determines the singular behaviour of $A(x)$!!!

Critical vs. Subcritical Graphs

$$A(x) = xC^\bullet(x),\ \Phi(x,a) = xe^{B^\bullet(x)},\ xC^\bullet(x) = xe^{B^\bullet(xC^\bullet(x))}$$

$$\Longrightarrow \quad \boxed{A(x) = \Phi(x, A(x))}$$

Case 1: the subcritical case. The system

$$a_0 = x_0 e^{B^\bullet(a_0)}, \quad 1 = x_0 e^{B^\bullet(x_0)} B^{\bullet\prime}(a_0)$$

has positive solutions x_0, a_0 such that a_0 is smaller than the radius of convergence η of $B^\bullet$. Equivalenty

$$\boxed{\eta B''(\eta) \in (1, \infty]}$$

Case 2: the critical case. The other case:

$$\boxed{\eta B''(\eta) = 1}.$$

Here the singular behaviour of $B^\bullet$ determines the singular behaviour of $C^\bullet(x)$.

Critical vs. Subcritical Graphs

- **Trees** are subcritical
- **Series-parallel graphs** are subcritical
- **Planar graphs** are **critical**

Lemma. If $B^\bullet(x)$ is entire or has a squareroot singularity:

$$B^\bullet(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\eta}},$$

then we are in the **subcritical** case.

Critical vs. Subcritical Graphs

What does "subcritical" mean?

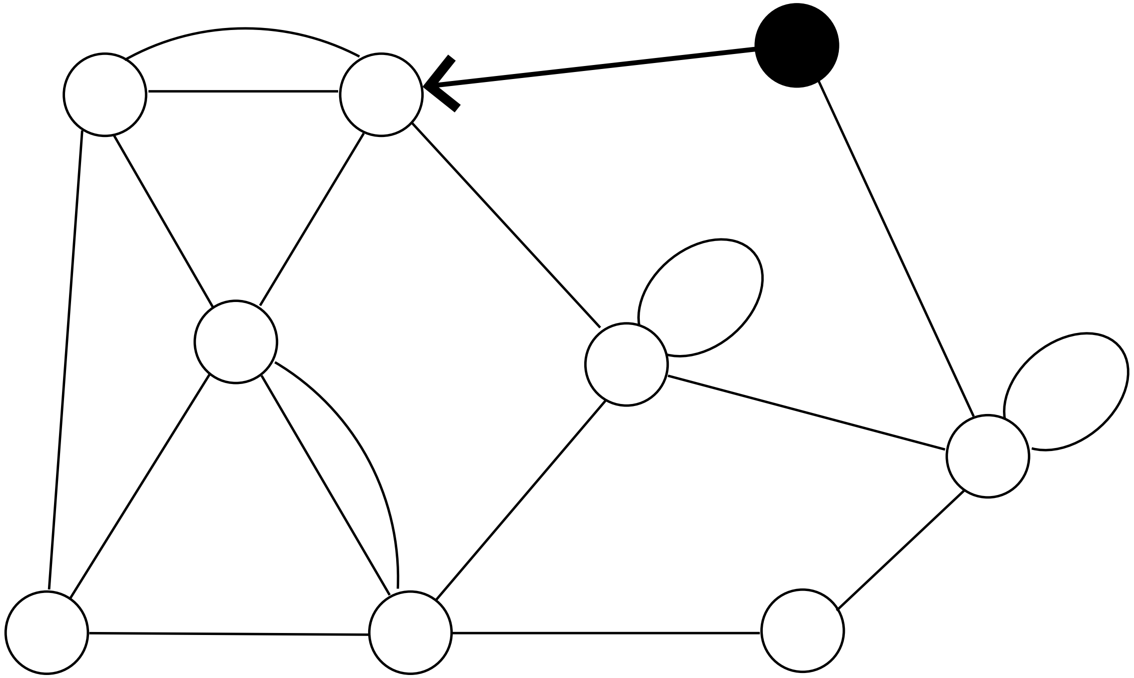
In a subcritical graph class the **average size of the 2-connected components is bounded**.

$\Longrightarrow$ This leads to a **tree like structure**.

$\Longrightarrow$ The **law of large numbers** should apply so that we can expect **universal behaviours** that are independent of the the precise structure of 2-connected components.

Critical **graph classes** are notoriously more difficult to analyze and we cannot expect universal laws.

Planar Maps



A **planar map** is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane.
A map is **rooted** if a vertex v and an edge e incident with v are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the **outer face**.

Planar Maps

M_n ... number of rooted maps with n edges [Tutte]

$$\boxed{M_n = \frac{2(2n)!}{(n+2)!n!}3^n}$$

The proof is given with the help of generating functions and the so-called **quadratic method**.

Asymptotics:

$$M_n \sim c \cdot n^{-5/2} 12^n$$

Planar Maps

Generating functions

$M_{n,k}$ ... number of maps with n edges and outer-face-valency k

$$M(z,u) = \sum_{n,k} M_{n,k} u^k z^n$$

$$\boxed{M(z,u) = 1 + zu^2 M(z,u)^2 + uz \frac{uM(z,u) - M(z,1)}{u-1}}$$

u ... **"catalytic variable"**

2-Connected Planar Maps

$B(z)$ ... GF of 2-connected rooted planar maps

$$\boxed{M(z) = B(zM(z)^2)}$$

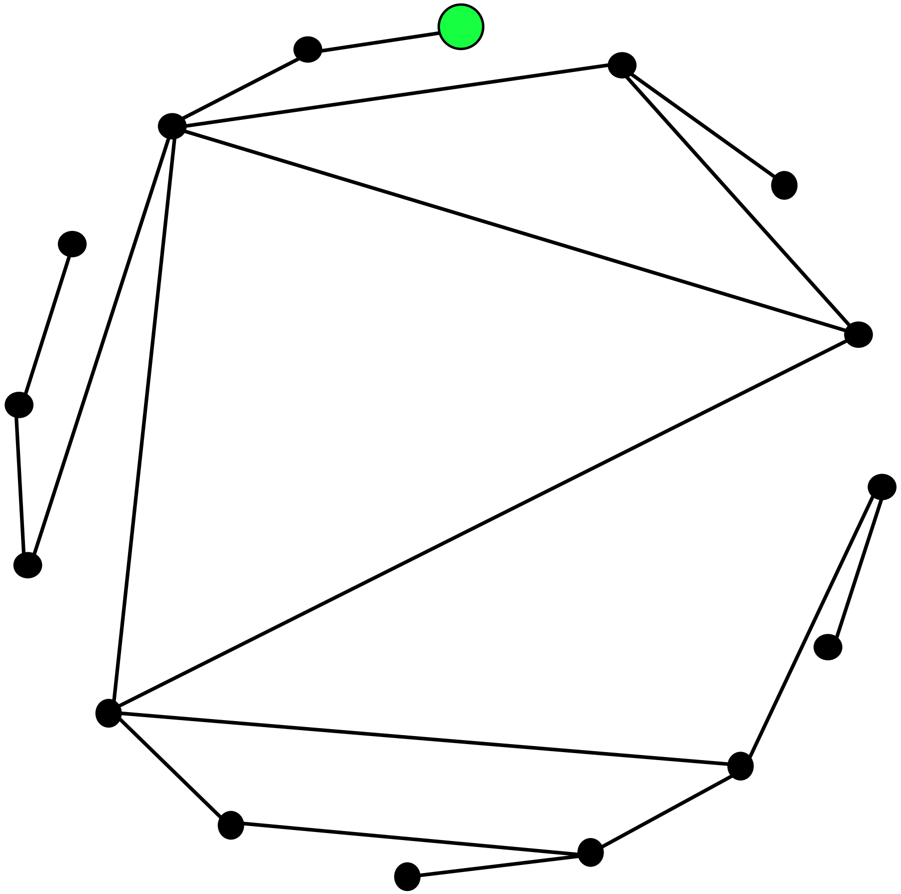
and

$$M(z,u) = B\left(zM(z)^2, \frac{uM(z,u)}{M(z)}\right)$$

Planar maps are also **critical**.

The equations are slightly different but analytically they are very similar.

Non-Crossing Configurations



Rooted convex n-gon with non-intersecting straight lines as edges
(we restrict ourselves to connected graphs)

Non-Crossing Configurations

$$\boxed{C(z) = \frac{z}{1 - B(C(z)^2/z)}}$$

$B(z)$ ... GF for 2-connected non-crossing configurations (dissections):

$$B(z) = z + \frac{B(z)^2}{1 - B(z)}$$

$$B(z) = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4}$$

Non-crossing configurations are **subcritical**

Unlabelled Graph Classes

Cycle index sums

$$Z_{\mathcal{G}}(s_1, s_2, \ldots) := \sum_n \frac{1}{n!} \sum_{\substack{\sigma, g \in \mathfrak{S}_n \times \mathcal{G}_n \\ \sigma \cdot g = g}} s_1^{c_1(\sigma)} s_2^{c_2(\sigma)} \cdots s_n^{c_n(\sigma)}$$

where $c_j(\sigma)$ denotes the number of cycles of size j in $\sigma \in \mathfrak{S}_n$

$$G(x) = Z_{\mathcal{G}}(x, x^2, x^3, \cdots)$$

$$Z_{\mathcal{G}^\bullet}(s_1, s_2, \ldots) = \frac{\partial}{\partial s_1} Z_{\mathcal{G}}(s_1, s_2, \ldots)$$

$$G^\bullet(x) = Z_{\mathcal{G}^\bullet}(x, x^2, x^3, \cdots) = \frac{\partial}{\partial s_1} Z_{\mathcal{G}}(x, x^2, x^3, \cdots)$$

Unlabelled Graph Classes

Block decomposition

$$G(x) = \exp\left(\sum_{i\geq 1} \frac{1}{i} C(x^i)\right)$$

$$\boxed{C^\bullet(x) = \exp\left(\sum_{i\geq 1} \frac{1}{i} Z_{B^\bullet}(x^i G^\bullet(x^i), x^{2i} G^\bullet(x^{2i}), \ldots)\right)}$$

• Dichotomy between **subcritical** and **critical** can be defined in a natural way.

• Unlabelled **trees** are **subcritical**.

• Unlabelled **series-parallel graphs** are **subcritical**.

Subcritical Graph Classes

Universal properties

- **Asymptotic enumeration**:

Labelled case:

$$\boxed{g_n \sim g\, n^{-5/2} \rho^{-n} n!}$$

Unlabelled case:

$$\boxed{g_n \sim g\, n^{-5/2} \rho^{-n}}$$

($g > 0$, ρ ... radius of convergence of $G(z)$)

[D.+Fusy+Kang+Kraus+Rue 2011]

Subcritical Graph Classes

Universal properties

• **Additive parameters** [D.+Fusy+Kang+Kraus+Rue 2011]

X_n ... number of **edges** / number of **blocks** / number of **cut-vertices** / number of **vertices of degree** k

Central limit theorem:

$$\boxed{\frac{X_n - \mu n}{\sqrt{n}} \to N(0, \sigma^2)}$$

with $\mu > 0$ and $\sigma^2 \geq 0$.

Remark. There is an easy to check "combinatorial condition" that ensures $\sigma^2 > 0$.

Subcritical Graph Classes

Proof Methods:
Refined versions of the functional equation $C^\bullet(x) = e^{B^\bullet(xC^\bullet(x))}$,
+ singularity analysis (**always squareroot singularity**)

E.g: number of edges:

$$C^\bullet(x,y) = e^{B^\bullet(xC^\bullet(x,y),y)}$$

$$\longrightarrow \quad C^\bullet(x,y) = g(x,y) - h(x,y)\sqrt{1 - \frac{x}{\rho(y)}}$$

$$\longrightarrow \quad [x^n]C^\bullet(x,y) \sim C(y)\rho(y)^{-n} n^{-3/2}$$

+ application of Quasi-Power-Theorem (by Hwang).

Subcritical Graph Classes

Universal properties in the labelled case

- **Maximum block size** $M_n^{(2)}$

$$\boxed{\mathbb{E}\, M_n^{(2)} = O(\log n)}$$

If the limit $\lim b_{n+1}/(nb_n)$ exists and is positive then $\mathbb{E}\, M_n^{(2)}$ is of order $\log n$ and the deviation from the mean is a disrete version of the Gumbel distribution.

- **Diameter** D_n

$$\boxed{c_1\sqrt{n} \le \mathbb{E}\, D_n \le c_2\sqrt{n \log n}}$$

- **Maximum degree** Δ_n

$$\boxed{c_1 \log n \le \mathbb{E}\, \Delta_n \le c_2 \log n}$$

Maximum Block Size

$B_k^\bullet(x)$ ... GF for 2-connected graphs of size $\le k$
$C_k^\bullet(x)$ ... GF for connected graphs of size $\le k$

$$\boxed{C_k^\bullet(x) = e^{B_k^\bullet(xC_k^\bullet(x))}}$$

$$\Longrightarrow \quad [x^n]C_k^\bullet(x) \sim c_k \rho_k^{-n} n^{-3/2}$$

with $\rho_k = \rho + O(\gamma^k)$ and $c_k = c + O(\gamma^k)$ for some $0 < \gamma < 1$.

$$\Longrightarrow \quad \mathbb{P}[M_n^{(2)} \le k] \sim \left(\frac{\rho}{\rho_k}\right)^n \ge e^{-Cn\gamma^k}$$

$$\Longrightarrow \quad \mathbb{E}\, M_n^{(2)} = O(\log n).$$

Diameter

Lower bound. $\underline{D}_n$ ... maximum number of blocks in a path

Tree structure $\Longrightarrow \mathbb{E}\,\underline{D}_n \sim c_1\sqrt{n}$

Upper bound. $\overline{D}_n$ ... maximum sum of block-heights on a path

$\overline{d}_n(v)$ ... sum of block-heights on path between v and the root

$Y_{n,h}$ ... profile related to $\overline{d}_n$: number of vertices with $\overline{d}_n(v) = h$

$L_h(x,u)$ ... GF corresponding to the profile $Y_{n,h}$

$B^{\bullet}_{=k}(x)$ ... GF of blocks with height $= k$

$$\boxed{L_h(x,u) = \exp\left(\sum_{k\le h} B^{\bullet}_{=k}(xL_{h-k}(x,u))\right)}$$

$M_h(x) = \frac{\partial}{\partial u} L_h(x,u)|_{u=1}$ ... GF of $\mathbb{E}\,Y_{n,k}$:

$$M_h(x) = e^{\sum_{k\le h} B^{\bullet}_{=k}(xC^{\bullet}(x))} \sum_{k\le h} {B^{\bullet}_{=k}}'(xC^{\bullet}(x)) M_{h-k}(x)$$

Diameter

$$\Longrightarrow \quad M_h(x) \sim C(x)\alpha(z)^h,$$

where $\alpha(z) = 1 - c'\sqrt{1 - x/\rho} + O(|x - x_0|)$

$$\mathbb{E}\, Y_{n,h} \sim c_1 h e^{-c_2 h^2/n}$$

First moment method: $\mathbb{P}[X > 0] \le \min\{1, \mathbb{E}\, X\}$

$$\mathbb{P}[\overline{D}_n > h] = \mathbb{P}[Y_{n,h} > 0] \le \min\{1, \mathbb{E}\, Y_{n,h}\}$$

$$\Longrightarrow \quad \mathbb{E}\, \overline{D}_n = \sum_{h \ge 0} \mathbb{P}[\overline{D}_n > h] = O(\sqrt{n \log n}).$$

Conclusion. $\underline{D}_n \le D_n \le \overline{D}_n$

$$\Longrightarrow \quad c_1\sqrt{n} \le \mathbb{E}\, D_n \le c_2\sqrt{n \log n}$$

Maximum Degree

Lower bound. $\underline{\Delta}_n$ ... maximum block degree of cut-vertices

Tree structure $\Longrightarrow \mathbb{E}\,\underline{\Delta}_n \sim c_1 \log n$

Upper bound. $D_n^{(r)}$ ... root degree

$B^\bullet(x,u)$ ... GF for root degree for 2-connected graphs

$C^\bullet(x,u)$ ... GF for root degree for connected graphs:

$$\boxed{C^\bullet(x,u) = e^{B^\bullet(xC^\bullet(x),u)}}$$

p_{nk} ... probability that the root vertex has degree k:

$$p_{n,k} = \frac{[x^n u^k]\, C^\bullet(x,u)}{[x^n]\, C^\bullet(x)}$$

Z_{nk} ... number vertices of degree k in connected graphs of size n

$$\mathbb{E}\, Z_{nk} = n p_{n,k}$$

Maximum Degree

First moment method: $\mathbb{P}[X > 0] \le \min\{1, \mathbb{E}\, X\}$

$$\begin{aligned}\mathbb{P}[\Delta_n > k] &= \mathbb{P}[Y_{n,k+1} + Y_{n,k+2} + \cdots > 0] \\ &\le \mathbb{E}\, Y_{n,k+1} + \mathbb{E}\, Y_{n,k+2} + \cdots \\ &= n(p_{n,k+1} + p_{n,k+2} + \cdots)\end{aligned}$$

$$\begin{aligned}[x^n u^k]\, C^\bullet(x,u) &\le [x^n] u^{-k} e^{B^\bullet(xC^\bullet(x),u)} \qquad (u > 1) \\ &\sim C(u) u^{-k} \rho^{-n} n^{-3/2}\end{aligned}$$

$$\Longrightarrow \quad p_{n,k} \le C(u)\, u^{-k} \qquad (u > 1)$$

$$\Longrightarrow \quad \mathbb{P}[\Delta_n > k] \le \min\{1,\, C\, n\, u^{-k}\}$$

$$\Longrightarrow \quad \mathbb{E}\, \Delta_n = \sum_{k \ge 0} \mathbb{P}[\Delta_n > k] = O(\log n).$$

Planar Maps

Additive Parameters

- $X_{n,k}$ ... number of vertices of degree k

$$\boxed{\frac{X_{n,k} - \mu_k n}{\sqrt{\sigma_k^2 n}} \to N(0,1)}$$

[D.+ Panagiotou, ANALCO 2012]

Planar Maps

Extremal Parameters

- **Maximum block size** $M_n^{(2)}$

$$\boxed{\mathbb{E}\, M_n^{(2)} \sim c_1 n}$$

with $c_1 = 1/3$ (GIANT 2-CONNECTED COMPONENT), Airy-law
[Gao+Wormald 1999, Banderier+Flajolet+Schaeffer+Soria 2001]

- **Diameter** D_n

$$\boxed{n^{\frac{1}{4}-\varepsilon} \le D_n \le n^{\frac{1}{4}+\varepsilon} \qquad w.h.p.}$$

[Chapuy+Fusy+Gimenez+Noy 2010]

- **Maximum degree** Δ_n

$$\boxed{\mathbb{E}\, \Delta_n \sim \log n}$$

+ discrete version of Gumbel law
[Gao+Wormald 2000]

Random Planar Graphs

Additive Parameters

- Y_n ... number of edges in a graph of size n

$$\boxed{\frac{Y_n - \mu n}{\sqrt{\sigma^2 n}} \to N(0,1)}$$

$\mu = 2.213...$, $\sigma^2 = 0.4303...$
[Gimenez+Noy 2009]

- $X_{n,k}$ ... number of vertices of degree k

$$\boxed{\mathbb{E}\, X_{n,k} \sim \mu_k n}$$

[D.+ Gimenez+ Noy 2011; Panagiotou+Steger 2011]

Open Problem. CLT ???

Remark. $(\mu_k)_k$ ... asymptotic degree distribution

Random Planar Graphs

Extremal Parameters

- **Maximum block size** $M_n^{(2)}$

$$\boxed{\mathbb{E}\, M_n^{(2)} \sim c_1 n}$$

with $c_1 = 0.959\ldots$ (GIANT 2-CONNECTED COMPONENT), Airy-law [Panagiotou+Steger 2010]

- **Diameter** D_n

$$\boxed{n^{\frac{1}{4}-\varepsilon} \le D_n \le n^{\frac{1}{4}+\varepsilon} \qquad w.h.p.}$$

[Chapuy+Fusy+Gimenez+Noy 2010]

- **Maximum degree** Δ_n

$$\boxed{\mathbb{E}\, \Delta_n \sim c \log n}$$

[D.+Gimenez+Noy+Panagiotou+Steger 2012+]

Random Planar Graphs

Degree Distribution (more precise formulation)

Theorem [D.+Giménez+Noy]

Let $p_{n,k}$ be the probability that a random node in a random planar graph $\mathcal{R}_n$ has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\ge 1} p_k w^k}$$

can be explicitly computed. We also have

$$p_k \sim c'\, k^{-\frac{1}{2}} q^k$$

for some $c' > 0$ and some $q < 1$.

Random Planar Graphs

Maximum Degree (more precise formulation)

Theorem [D.+Giménez+Panagiotou+Noy+Steger]

Set $c = (\log(1/q))^{-1} = 2.529464248...$, where $q = 0.6734506...$ appear in the asymptotics of $p_k \sim c' \, k^{-\frac{1}{2}} q^k$.

Then

$$\boxed{|\Delta_n - c \log n| = O(\log\log n)} \qquad w.h.p$$

and

$$\mathbb{E}\, \Delta_n \sim c \log n.$$

Remark. [McDiarmid+Reed (2008)]

$$\boxed{c_1 \log n \le \Delta_n \le c_2 \log n} \qquad w.h.p.$$

Maximum Degree

Relation to number of vertices of given degree

$X_n^{(k)}$ ... **number of vertices of degree** k in G_n.

$X_n^{(>k)} = X_n^{(k+1)} + X_n^{(k+2)} + \cdots$ ... number of vertices of degree $> k$.

Δ_n ... maximum degree:

$$\boxed{\Delta_n > k \iff X_n^{(>k)} > 0}$$

First moment method:

$$\begin{aligned}\mathbb{P}\{\Delta_n > k\} &= \mathbb{P}\{X_n^{(>k)} > 0\} \\ &\le \min\{1, \mathbb{E}\, X_n^{(>k)}\}\end{aligned}$$

Maximum Degree

First moments

$p_{n,k}$ ... **probability that a random vertex** in G_n **has degree** k

$$\boxed{\mathbb{E}\, X_n^{(k)} = n\, p_{n,k}}$$

$$\Longrightarrow \quad \mathbb{E}\, X_n^{(>k)} = \mathbb{E}\left(\sum_{\ell > k} X_n^{(\ell)} \right) = n \sum_{\ell > k} p_{n,\ell}.$$

Precise asymptotics or upper bounds for $p_{n,k}$ are needed that are **uniform in** n **and** k.

Maximum Degree

Remark 1 In order to get upper bound it is sufficient to know

$$p_{n,k} = O(q^k) \quad \text{uniformly for all } n, k \geq 0$$

for some q.

Proof Strategy

1. Establish generating functions for $p_{n,k}$

2. Analytic structure of generating functions

3. Upper bound with First Moment Method

4. Lower bound with Boltzmann Sampling

Random Planar Graphs

Counting Generating Functions

$$
\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right), \\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right), \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y}, \\
\frac{M(x,D)}{2x^2D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}, \\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right), \\
U &= xy(1+V)^2, \\
V &= y(1+U)^2.
\end{aligned}
$$

Random Planar Graphs

Asymptotic enumeration of planar graphs

$$b_n = b \cdot \rho_1^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$c_n = c \cdot \rho_2^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$g_n = g \cdot \rho_2^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$\rho_1 = 0.03819...,$$

$$\rho_2 = 0.03672841...,$$

$$b = 0.3704247487... \cdot 10^{-5},$$

$$c = 0.4104361100... \cdot 10^{-5},$$

$$g = 0.4260938569... \cdot 10^{-5}$$

Random Planar Graphs

Generating functions for the degree distribution of planar graphs

$C^\bullet = \frac{\partial C}{\partial x}$ ... GF, where one vertex is marked

w ... additional variable that *counts* the **degree of the marked vertex**

Generating functions:

$G^\bullet(x, y, w)$	**all rooted** planar graphs
$C^\bullet(x, y, w)$	**connected rooted** planar graphs
$B^\bullet(x, y, w)$	**2-connected rooted** planar graphs
$T^\bullet(x, y, w)$	**3-connected rooted** planar graphs

Random Planar Graphs

$$
\begin{aligned}
G^\bullet(x,y,w) &= \exp\left(C(x,y,1)\right) C^\bullet(x,y,w), \\
C^\bullet(x,y,w) &= \exp\left(B^\bullet\left(xC^\bullet(x,y,1),y,w\right)\right), \\
w\frac{\partial B^\bullet(x,y,w)}{\partial w} &= xyw \exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)} T^\bullet\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) \\
D(x,y,w) &= (1+yw)\exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)} \times\right. \\
&\qquad \left. \times\, T^\bullet\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) - 1 \\
S(x,y,w) &= xD(x,y,1)\left(D(x,y,w) - S(x,y,w)\right), \\
T^\bullet(x,y,w) &= \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy} + \frac{1}{1+xy} - 1 - \right. \\
&\qquad \left. - \frac{(u+1)^2\left(-w_1(u,v,w) + (u-w+1)\sqrt{w_2(u,v,w)}\right)}{2w(vw+u^2+2u+1)(1+u+v)^3}\right), \\
u(x,y) &= xy(1+v(x,y))^2, \quad v(x,y) = y(1+u(x,y))^2.
\end{aligned}
$$

Degree Distribution

with polynomials $w_1 = w_1(u, v, w)$ and $w_2 = w_2(u, v, w)$ given by

$$\begin{aligned} w_1 =& -uvw^2 + w(1 + 4v + 3uv^2 + 5v^2 + u^2 + 2u + 2v^3 + 3u^2v + 7uv) \\ &+ (u+1)^2(u + 2v + 1 + v^2), \end{aligned}$$

$$\begin{aligned} w_2 =& u^2v^2w^2 - 2wuv(2u^2v + 6uv + 2v^3 + 3uv^2 + 5v^2 + u^2 + 2u + 4v + 1) \\ &+ (u+1)^2(u + 2v + 1 + v^2)^2. \end{aligned}$$

Asymptotics for Random Planar Graphs

Functional equations

Suppose that $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, 1, a_0), \quad 1 = \Phi_a(x_0, 1, a_0)}.$$

Then there exists analytic function $g(x,u), h(x,u)$, and $\rho(u)$ such that locally

$$\boxed{A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

Asymptotics for Random Planar Graphs

Asymptotics for coefficients

$$A(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\rho}} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad \boxed{[x^n]\, A(x) = \frac{h(\rho)}{2\sqrt{\pi}}\, \rho^{-n}\, n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right)}.$$

Similarly:

$$A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad [x^n]\, A(x,u) = \frac{h(\rho(u),u)}{2\sqrt{\pi}}\, \rho(u)^{-n}\, n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Asymptotics for Random Planar Graphs

Asymptotics for coefficients

and

$$A(x) = g(x) + h(x)\left(1 - \frac{x}{\rho}\right)^{\alpha} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad [x^n]\, A(x) = \frac{h(\rho)}{\Gamma(-\alpha)}\, \rho^{-n}\, n^{-\alpha-1}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Asymptotics for Random Planar Graphs

Singular expansion

$$\begin{aligned} A(x) &= \boxed{g(x) - h(x)\sqrt{1-\frac{x}{\rho}}} \\ &= \left(g_0 + g_1(x-\rho) + g_2(x-\rho)^2 + \cdots\right) \\ &\quad + \left(h_0 + h_1(x-\rho) + h_2(x-\rho)^2 + \cdots\right)\sqrt{1-\frac{x}{\rho}} \\ &= a_0 + a_1\left(1-\frac{x}{\rho}\right)^{\frac{1}{2}} + a_2\left(1-\frac{x}{\rho}\right)^{\frac{2}{2}} + a_3\left(1-\frac{x}{\rho}\right)^{\frac{3}{2}} + \cdots \\ &= \boxed{a_0 + a_1X + a_2X^2 + a_3X^3 + \cdots} \end{aligned}$$

with

$$X = \sqrt{1-\frac{x}{\rho}}.$$

Asymptotics for Random Planar Graphs

$$U(x,y) = xy(1+V(x,y))^2,$$
$$V(x,y) = y(1+U(x,y))^2$$
$$\Longrightarrow \quad U(x,y) = xy(1+y(1+U(x,y))^2)^2$$
$$\Longrightarrow \quad \boxed{U(x,y) = g(x,y) - h(x,y)\sqrt{1-\frac{y}{\tau(x)}}}$$
$$\Longrightarrow \quad V(x,y) = g_2(x,y) - h_2(x,y)\sqrt{1-\frac{y}{\tau(x)}}$$

$$M(x,y) = x^2y^2\left(\frac{1}{1+xy}+\frac{1}{1+y}-1-\frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right)$$
$$!!! \quad \Longrightarrow \quad \boxed{M(x,y) = g_3(x,y) + h_3(x,y)\left(1-\frac{y}{\tau(x)}\right)^{\frac{3}{2}}}$$

due to cancellation of the $\sqrt{1-y/\tau(x)}$-term

Asymptotics for Random Planar Graphs

$$\frac{M(x,D)}{2x^2D} = \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}$$

!!! $\Longrightarrow$ $\boxed{D(x,y) = g_4(x,y) + h_4(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}}}$

due to interaction of the singularities!!!

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

!!! $\Longrightarrow$ $\boxed{B(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{5}{2}}}$

$\Longrightarrow$ $\boxed{b_n \sim b \cdot R(1)^{-n} n^{-\frac{7}{2}} n!}$

Asymptotics for Random Planar Graphs

$$B'(x,y) = g_6(x,y) + h_6(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}},$$

$$C'(x,y) = e^{B'(xC'(x,y),y)},$$

!!! $\Longrightarrow$ $\boxed{C'(x,y) = g_7(x,y) + h_7(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{3}{2}}}$

due to interaction of the singularities!!!

$$\Longrightarrow \quad \boxed{C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}}$$

$$\Longrightarrow \quad \boxed{c_n \sim c\, r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotics for Random Planar Graphs

$$C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}$$

$$\Longrightarrow \quad G(x,y) = e^{C(x,y)} = g_9(x,y) + h_9(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}.$$

$$\Longrightarrow \quad \boxed{g_n \sim g \cdot r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotic Degree Distribution

3-connected planar graphs

$$T^\bullet(x,y,w) = \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy} + \frac{1}{1+xy} - 1 - \right.$$
$$\left. - \frac{(U+1)^2\left(-w_1(U,V,w) + (U-w+1)\sqrt{w_2(U,V,w)}\right)}{2w(Vw+U^2+2U+1)(1+U+V)^3}\right),$$

$$\tilde{u}_0(y) = -\frac{1}{3} + \sqrt{\frac{4}{9} + \frac{1}{3y}}, \quad r(y) = \frac{\tilde{u}_0(y)}{y(1+y(1+\tilde{u}_0(y))^2)^2},$$

$$\tilde{X} = \sqrt{1 - \frac{x}{r(y)}}$$

$$\Longrightarrow \quad \boxed{T^\bullet(x,y,w) = \tilde{T}_0(y,w) + \tilde{T}_2(y,w)\tilde{X}^2 + \tilde{T}_3(y,w)\tilde{X}^3 + O(\tilde{X}^4)}$$

due to cancellation of the $\sqrt{1-x/r(z)}$-term.

Asymptotic Degree Distribution

Planar networks

$$D(x,y,w) = (1+yw)\exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)} \times \right.$$
$$\left. \times\, T^{\bullet}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) - 1$$
$$S(x,y,w) = xD(x,y,1)\left(D(x,y,w) - S(x,y,w)\right)$$

$$\tau(x) \ \ldots \ \text{inverse function of } r(y)$$

$$D(R(y),y,1) = \tau(R(y))$$

$$X = \sqrt{1 - \frac{x}{R(y)}}$$

$$\Longrightarrow \quad \boxed{D(x,y,w) = D_0(y,w) + D_2(y,w)X^2 + D_3(y,w)X^3 + O(X^4)},$$

Asymptotic Degree Distribution

2-connected planar graphs

$$w\frac{\partial B^\bullet(x,y,w)}{\partial w} = xyw\,\mathrm{exp}\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)$$

$$\Longrightarrow \quad \boxed{B^\bullet(x,y,w) = B_0(y,w)+B_2(y,w)X^2+B_3(y,w)X^3+O(X^4)}$$

Remark. All these functions $B_j(y,w)$ can be *explicitly* computed.

If $x=\rho_B$ then they are analytic for $w<w_0$ and have an algebraic singularity at $w=w_0$!!!

Asymptotic Degree Distribution

connected planar graphs

$$C^\bullet(x, 1, w) = \exp\left(B^\bullet\left(xC'(x), 1, w\right)\right)$$

$$\Longrightarrow \quad \boxed{C^\bullet(x, y, w) = C_0(y, w) + C_2(y, w)X^2 + C_3(y, w)X^3 + O(X^4)}$$

$$X = \sqrt{1 - \frac{x}{R(y)}}$$

Asymptotic Degree Distribution

connected planar graphs

$$p_{n,k} = \frac{[x^n w^k] C^\bullet(x,1,w)}{[x^n] C^\bullet(x,1,1)}$$

$$[x^n] C^\bullet(x,1,1) \sim c_1 n^{-5/2} \rho_C^{-n}.$$

$$\begin{aligned}[x^n w^k] C^\bullet(x,1,w) &\le w_0^{-k} [x^n] C^\bullet(x,1,w_0) \\ &\sim w_0^{-k} c_2 n^{-5/2} \rho_C^{-n}\end{aligned}$$

$$\Longrightarrow \boxed{p_{n,k} = O(w_0^{-k}) = O(q^k)} \qquad (q = 1/w_0)$$

Remark. Here we use $\boxed{x \mapsto C^\bullet(x, w_0)}$

Asymptotic Degree Distribution

First moment method for upper bound

$$\Longrightarrow \boxed{\mathbb{P}\{\Delta_n > k\} = O(nq^k)}$$

$$\Longrightarrow \boxed{\mathbb{P}\{\Delta_n \le c \log n + r\} \le 1 - O(q^r)}$$

Boltzmann Sampling

Probability distribution

$C^\bullet(x)$ ... (exponential) generating function for rooted (connected) planar graphs

γ ... (random) rooted connected planar graph

Boltzmann distribution

$$\mathsf{Pr}_x[\gamma] = \frac{x^{|\gamma|}}{|\gamma|!\, C^\bullet(x)}$$

Special case: $x = \rho_C$

$$\mathsf{Pr}[\gamma] = \frac{\rho_C^{|\gamma|}}{|\gamma|!\, C^\bullet(\rho_C)}$$

Boltzmann Sampling

Conditional distribution

$$\Pr[|\gamma| = n] = \frac{c_n^{\bullet} \rho_C^n}{|\gamma|!\, C(\rho_C)}$$

$$\Pr[\gamma \,|\, |\gamma| = n] = \frac{1}{c_n^{\bullet}}$$

rd ... root degree

$$\Pr[rd(\gamma) = k] = \frac{[w^k] C^{\bullet}(\rho_C, w)}{C^{\bullet}(\rho_C)}$$

Boltzmann Sampling

Root degree distribution

Lemma 1

$$\Pr[\mathrm{rd}(\gamma) \geq k] \sim c_3 k^{-5/2} w_0^{-k}$$

Proof.

$$C^\bullet(\rho_C, w) = C_0(1, w) = g(w) + h(w)\,(1 - w/w_0)^{3/2}$$

Remark. Here we use the function $\boxed{w \mapsto C^\bullet(\rho_C, w)}$

Boltzmann Sampling

Largest 2-connected component

Lemma 2

lb ... size of largest 2-connected component

$$\mathbb{P}\{\mathrm{lb}(C_n) = \lfloor (1-\rho_B)B''(\rho_B)n + xn^{2/3} \rfloor\} = \Theta(n^{-2/3})$$

uniformly for $|x| \le C$ (for a given constant C).

Boltzmann Sampling

Largest 2-connected component

Lemma 3

Suppose that $|m-(1-\rho_B)B''(\rho_B)n| \le Cn^{2/3}$ and $\gamma_1,\ldots,\gamma_m$ random rooted connected planar graphs (drawn according to the Boltzmann distribution). Then

$$\Pr\left[\sum_{i=1}^{n} |\gamma_i| = n\right] = \Theta(n^{-2/3}).$$

Boltzmann Sampling

Completion of the proof

B ... largest 2-connected component of random connected planar graph
m ... size of B: $|m-(1-\rho_B)B''(\rho_B)n| \le Cn^{2/3}$ w.h.p.
$\gamma_1,\ldots,\gamma_m$ ... connected graph rooted at vertices of B:

$$\Delta_n \ge \max_{1\le j\le m} \mathrm{rd}(\gamma_j)$$

W.h.p. $\gamma_1,\ldots,\gamma_m$ can be drawn independently according to the Boltzmann distribution: Lemma 1 $\Longrightarrow$

$$\Pr\left[\max_{1\le j\le m} \mathrm{rd}(\gamma_j) < k\right] \le \left(1-c_3k^{-5/2}w_0^{-k}\right)^m$$

Boltzmann Sampling

Completion of the proof

$$\Pr\left[\max_{1\le j\le m} \mathrm{rd}(\gamma_j) < k\right] \le \left(1 - c_3 k^{-5/2} w_0^{-k}\right)^m$$

$k = (1-\delta)\log_{w_0} n = c(1-\delta)\log n$,
where $\delta = C \log\log n / \log n$;
$m \ge n/2$ (w.h.p.)

$$\Longrightarrow \quad \Pr\left[\max_{1\le j\le m} \mathrm{rd}(\gamma_j) < k\right] = O\left(e^{-c_4 (\log n)^{C-5/2}}\right)$$

$$\Longrightarrow \quad \boxed{\mathbb{P}\{\Delta_n \ge c(1-\delta)\log n\} \ge 1 - O\left(e^{-c_4 (\log n)^{C-5/2}}\right)}$$

Thank You for Your Attention!